9. Moduli spaces

9.1 Overview

Various surfaces with tilings that look the same topologically may not be isometric. In fact, there are infinite families of topologically similar, non-isometric, tilings depending on continuous parameters called *moduli.* We have not yet seen this phenomenon since it cannot occur for tilings by triangles. To better understand moduli we need to introduce the real and complex moduli spaces of curves of genus σ,and then we will study some particular low dimensional sets of these spaces coming from quadrilateral tilings. First we need some definitions.

Definition 9.1 Two surfaces S, S' are conformally equivalent is there is a biholomorphic map $h : S \to S'$ (h is bijective and h and h^{-1} are both holomorphic.). The (suitably topologized) set of all conformal equivalence classes of surfaces of genus σ, $\mathcal{M}_\sigma = \mathcal{M}_\sigma(\mathbb{C})$, is called the moduli space of surfaces of genus σ.

Definition 9.2 A *symmetry* of a surface is an anti-conformal involution. If the symmetry has a fixed point it is called a *reflection*. A surface is called *real* if it is conformally equivalent to a Riemann surface defined by algebraic equations with real coefficients. The moduli space of real surfaces $\mathcal{M}_\sigma(\mathbb{R})$ is the set of real surfaces in the moduli space $\mathcal{M}_\sigma$.

Remark 9.1 A Riemann surface is S is real if and only if one of the following holds:

- The surface has a symmetry.
- $S = \mathbb{H}/\Gamma$ where $\Gamma = s\Gamma s^{-1}$ for some reflection or glide reflection on $\mathbb{H}$.

  Every surface with a kaleidoscopic tiling is contained in $\mathcal{M}_\sigma(\mathbb{R})$, though $\mathcal{M}_\sigma(\mathbb{R})$ contains other points as well.

The moduli space $\mathcal{M}_\sigma$ is a complicated algebraic variety which is not completely understood. Its dimension as a complex variety is $3\sigma - 3$, if $\sigma \geq 2$. The real moduli space has the same real dimension. We may also consider the space of conformal equivalence classes of surfaces with n distinguished, ordered points. The corresponding moduli space is of complex dimension $3\sigma - 3 + n$, if $\sigma \geq 2$, or $\sigma = 1$, $n \geq 1$, or $\sigma = 0$, $n \geq 3$.

Obviously, since conformally equivalent surfaces are isometric, they have isomorphic automorphism groups, and have the same types of tilings. Therefore, the variation in tilings need to be considered on the (real) moduli space. In fact the set of surfaces with topologically equivalent tilings form nice subsets of the moduli space, defined by their automorphism group or their symmetry type. Indeed, the moduli space may be decomposed into a disjoint union of smooth complex manifolds or subvarieties

$$\mathcal{M}_\sigma = \bigcup_G \mathcal{M}_\sigma^G$$

where $\mathcal{M}_\sigma^G$ is a the subset of surfaces whose automorphism group is G. This decomposition of the moduli space is called the equisymmetric stratification. The strata $\mathcal{M}_\sigma^G$ are called equisymmetric strata. A precise definition of this stratification and description of the groups G is given in [6]. The zero dimensional strata in the decomposition, i.e., the points, correspond to tiling by triangles. Tilings by quadrilaterals and their complex extensions give rise to one-dimensional strata. Pentagonal tilings and their complex extensions give rise to two-dimensional strata and so on. The largest stratum which is an open subset corresponding to surfaces with no non-trivial automorphism.

9.2 Moduli of quadrilateral rotation groups

For some of the background in this section see [2] and [7].

We now examine the case of moduli of quadrilateral tilings in some detail. Let S be a surface with a kaleidoscopic tiling by (k, l, m, n)-quadrilaterals. The quotient space $T = S/G$, by the conformal tiling group, is a sphere and $q : S \to T$ is branched over 4 points. Now as $G \lhd G^*$ then $\langle \phi \rangle = G^*/G$ acts on T. By considering the local action of a reflection on S, we see that ϕ must be an orientation-reversing isometry of the sphere, with fixed points, and hence is a reflection. Moreover, the fixed points of ϕ must contain the four distinct branch points of orders k, l, m, and n. For the branch points are the images of vertices and each of them lies on a mirror. All the mirrors project to the fixed point subset of ϕ.If we consider more generally a surface S with a (k, l, m, n)-G-action then the four branch points could be arbitrarily placed on the sphere. Using this idea of a general quotient space we will outline a general solution to the problem in five steps

1. Rewrite the classification problem as a covering space problem $q : S \to T = S/G$, and include complex moduli.
2. Determine the moduli space of ordered and unordered quotient surfaces $T = S/G$.
3. Develop a calculus of determining exactly which conformally inequivalent surfaces lie over a given quotient surface T.
4. Determine the moduli space of surfaces with (k, l, m, n)-G-action as a covering of the moduli space of quotients.
5. Determine the tilings by quadrilaterals as a component of the real moduli subspace.

Complex moduli and the moduli of quotients Let w_1, w_2, w_3, and w_4 be the four branch points of $S \to T = S/G$ chosen so that the branching orders are k, l, m, and n, respectively. There is a unique linear fractional transformation

$$L(z) = \frac{(z - w1)(w_2 - w_3)}{(z - w3)(w_2 - w_1)}$$

such that

$$L(w_1) = 0, L(w_2) = 1, L(w_3) = \infty.$$

The value $L(w_4) = \chi(w_1, w_2, w_3, w_4)$ is the well-known cross ratio of four points, which we denote by the moduli parameter λ and is given by

$$\lambda = \chi(w_1, w_2, w_3, w_4) = L(w_4) = \frac{(w_4 - w_1)(w_2 - w_3)}{(w_4 - w_3)(w_2 - w_1)}.$$

Since the four points in $\{0, 1, \infty, \lambda\}$ must be distinct, λ cannot take on any of the values $0, 1, \infty$. As proven below, given λ, there is a corresponding surface S' and an action of G on S' such that $S' \to S'/G$ is branched over $0, 1, \infty$, and λ. In, addition there is a G-equivariant conformal map $h : S \to S'$ covering L, i.e., the diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & S' \\ \downarrow & & \downarrow \\ T & \xrightarrow{L} & T \end{array}$$

commutes and $h(gz) = gh(z)$ for all $g \in G$. Thus in trying to understand moduli of the four branch point surfaces with G action we may make the assumption that the branch points are $0, 1, \infty$, and λ, and that $\lambda \in \mathbb{C} - \{0, 1\}$.

Notation 9.3 Let T_λ denote the surface $\mathbb{S}^2$ with the selection of four ordered distinct points $(w_1, w_2, w_3, w_4) = (0, 1, \infty, \lambda)$, $T_\lambda^\circ = \mathbb{S}^2 - \{0, 1, \infty, \lambda\} = \mathbb{C} - \{0, 1, \lambda\}$.

If any two of the k, l, m, n are equal it is possible that two different different moduli λ and λ' could have been chosen by taking a different ordering of the branch points, preserving branching order. In this case there is a fractional linear transformation of the Riemann sphere taking the set $\{0, 1, \infty, \lambda\}$ to $\{0, 1, \infty, \lambda'\}$ and preserving branch orders. Indeed if $(k, l, m, n) = (2, 2, 3, 6)$ then (w_1, w_2, w_3, w_4) and (w_2, w_1, w_3, w_4) would be the two possible choices. From

$$\lambda = \frac{(w_4 - w_1)(w_2 - w_3)}{(w_4 - w_3)(w_2 - w_1)}, \lambda' = \frac{(w_4 - w_2)(w_1 - w_3)}{(w_4 - w_3)(w_1 - w_2)}$$

We easily calculate that $\lambda' = 1 - \lambda$. The transformation $h : z \to 1 - z$ maps the ordered quadruple $(0, 1, \infty, \lambda)$ to $(1, 0, \infty, 1 - \lambda)$ and consequently $h : T_\lambda^\circ \to T_{1-\lambda}^\circ = T_{\lambda'}^\circ$ is a conformal map. The set of such transformations for various permutations is given Table 9.3 later in this section.

Proposition 9.1 *Let $S \to T = S/G$ where S has a (k, l, m, n)-action by G. Then we may assume that the branch points, taken in order, are $\{0, 1, \infty, \lambda\}$ for some $\lambda \in \mathbb{C} -$*

$\{0,1\}$. *Furthermore, if the branching data (k,l,m,n) has a symmetry $\pi \in \Sigma_4$ then an alternative branch set $\{0,1,\infty,\lambda'\}$ with $\lambda' = \ell_\pi(\lambda)$ per Table 9.3 below may occur. The set of all possible λ for a given symmetry type have the form $\{\ell_\pi(\lambda) : \ell_\pi \in F\}$ for some finite group F. The moduli space of unordered quotients is then $(\mathbb{S}^2 - \{0,1,\infty\})/F$. The groups F and a rational quotient map $p : \mathbb{S}^2 - \{0,1,\infty\} \to (\mathbb{S}^2 - \{0,1,\infty\})/F$ are given in Table 9.1 below.*

Table 9.1 - Moduli Spaces of Quotients.

branching data (k,l,m,n, distinct)	F	$(\mathbb{S}^2 - \{0,1,\infty\})/F$
(k,l,m,n)	$\{\lambda\}$	$\mathbb{C} - \{0,1\}$
$(k,k,l,m),(k,k,l,l),(k,l,m,m)$	$\{\lambda, 1-\lambda\}$	$\mathbb{C} - \{0\}$
(k,l,l,m)	$\{\lambda, \lambda/(\lambda-1)\}$	$\mathbb{C} - \{0\}$
$(k,k,k,l),(k,k,k,k)$	$\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$	$\mathbb{C}$

$p : \mathbb{S}^2 - \{0,1,\infty\} \to (\mathbb{S}^2 - \{0,1,\infty\})/F$

branching data (k,l,m,n, distinct)	p	degree	finite branch points of q
(k,l,m,n)	λ	1	none
$(k,k,l,m),(k,k,l,l),(k,l,m,m)$	$\lambda - \lambda^2$	2	1/4
(k,l,l,m)	$\lambda^2/(\lambda-1)$	2	4
$(k,k,k,l),(k,k,k,k)$	$\frac{(\lambda^2-\lambda+1)^2}{\lambda^2(1-\lambda)^2}$	6	$\frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$

Surfaces over the quotients We will now describe a method for determining the conformally inequivalent surfaces with (k,l,m,n)-G-actions lie over a given quotient space. Let T_λ be as previously defined and let S_λ be some surface with $(k,l,m,n)-G$-action such that $S_\lambda \to S_\lambda/G$ is branched over $\{0,1,\infty,\lambda\}$. Let $S_\lambda^\circ = q^{-1}(T_\lambda^\circ)$ be S_λ minus the inverse image of the branch points $\{0,1,\infty,\lambda\}$. It follows that $q : S_\lambda^\circ \to T_\lambda^\circ$ is an unbranched covering space with Galois group G. If $z_0 \in T_\lambda^\circ$ and $x_0 \in S_\lambda^\circ$ lies above z_0, then there is a surjective map $\eta_0 : \pi_1(T_\lambda^\circ, z_0) \to G$, with kernel $\pi_1(S_\lambda^\circ, x_0)$, as described in Proposition **??**. If another point $x_1 = gx_0$ lying above z_0 is chosen then the two maps η_0, η_1 so determined, are related by $\eta_1(\alpha) = g\eta_0(\alpha)g^{-1}$ as in equation 6.7 of Chapter 6. Thus if $\omega \in \mathrm{Aut}(G)$ is the automorphism $\omega = \mathrm{Ad}_g : h \to ghg^{-1}$ then $\eta_1 = \omega \circ \eta_0$, and the two maps have the same kernel.

On the other hand if $\omega \in \mathrm{Aut}(G)$ is arbitrary then $\omega \circ \eta_0$ and η_0 have the same kernel and determine the same cover. $S_\lambda^\circ \to T_\lambda^\circ$. Now suppose that $\eta' : \pi_1(T_\lambda^\circ, z_0) \to G$, is an arbitrary surjective map, and $q' : S_\lambda'^\circ \to T_\lambda^\circ$ is the covering space determined by $\ker \eta'$. Then a G action is determined on $S_\lambda'^\circ$ by

$$(g,x) \to (\eta \circ (\eta')^{-1}(g))x$$

where $\eta : \pi_1(T_\lambda^\circ, z_0) \to \mathrm{Gal}(S_\lambda^\circ/T_\lambda^\circ)$. The Riemann removable singularities theorem implies that $q' : S_\lambda'^\circ \to T_\lambda^\circ$ can be completed to a holomorphic map $S_\lambda' \to T_\lambda$ and that each element of $\mathrm{Gal}(S_\lambda^\circ/T_\lambda^\circ)$ can be completed to a biholomorphic map in $\mathrm{Gal}(S_\lambda/T_\lambda)$.

We use the notation S'_λ to indicate that this Riemann surface may not be conformally equivalent to S_λ. Thus for each $\mathrm{Aut}(G)$-class of homomorphisms $\pi_1(T^\circ_\lambda, z_0) \to G$ we get a unique S_λ and $(k,l,m,n) - G$-action such that $S_\lambda \to S_\lambda/G = T_\lambda$ is branched over $\{0,1,\infty,\lambda\}$.

Exercise 9.1 Show that the above is independent $z_0 \in T^\circ_\lambda$. Suggestion if δ is a path from z_0 to z_1 then $\mathrm{Ad}_\delta : \alpha \to \delta * \alpha * \delta^{-1}$ is an isomorphism $\pi_1(T^\circ_\lambda, z_0) \to \pi_1(T^\circ_\lambda, z_1)$, unique up to inner automorphism.

Next we answer: When do two branched covers $S_{\lambda_1} \to T_{\lambda_1}$ and $S_{\lambda_2} \to T_{\lambda_2}$ give conformally equivalent surfaces? First assume that $\lambda_1 \neq \lambda_2$. We classified previous exactly when and S_λ could have two different quotients T_{λ_1} and T_{λ_2}. Now suppose we have any conformal map $h : T^\circ_{\lambda_1} \to T^\circ_{\lambda_2}$, When can we fill in top row so that the diagram commutes.

$$\begin{array}{ccc} S^\circ_{\lambda_1} & \xrightarrow{\widetilde{h}} & S^\circ_{\lambda_2} \\ q_1 \downarrow & & \downarrow q_2 \\ T^\circ_{\lambda_1} & \xrightarrow{h} & T^\circ_{\lambda_2} \end{array}$$

According to the lifting criterion in Chapter 6 we must have $h_* q_{1*}(\pi_1(S^\circ_{\lambda_1}, x_1)) = q_{2*}(\pi_1(S^\circ_{\lambda_2}, x_2))$ for appropriately chosen x_1 and x_2. Alternatively if $\eta_1 : \pi_1(T^\circ_{\lambda_1}, z_0) \to G$, and $\eta_2 : \pi_1(T^\circ_{\lambda_2}, z_1) \to G$ define the two covers then there must exist an $\omega \in \mathrm{Aut}(G)$ such that

$$\eta_2 = \omega \circ \eta_1 \circ h_*^{-1}. \tag{1}$$

Thus we have the following Proposition.

Proposition 9.2 *Suppose that T_{λ_1} and T_{λ_2} are conformally equivalent with $\lambda_2 = \ell_\pi(\lambda_1)$ per table 9.2, but $\lambda_1 \neq \lambda_2$ Then 9.1 gives a 1-1 correspondence between the conformal equivalence classes of surfaces with (k,l,m,n)-G-action lying above $T^\circ_{\lambda_1}$ and $T^\circ_{\lambda_2}$.*

Generating vectors and conformal equivalence Next we need to finitize the determination of the $\mathrm{Aut}(G)$-classes of epimorphisms $\eta : \pi_1(T^\circ_\lambda, z_0) \to G$. Let $(w_1, w_2, w_3, w_4) = (0, 1, \infty, \lambda)$ and select a marking T°_λ as in Chapter **??**. Specifically,select a set of four loops $\gamma_1, \ldots, \gamma_4$ as follows:

- The loop γ_1 follows a path from z_0 almost to w_i travels in the counter-clockwise direction along a small circle centered a w_i and returns to z_0 along the same path.
- The paths are chosen and the circles are made so small that the only point of intersection among the loops is at z_0.
- The paths emanate from z_0 in the cyclic order $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ awe move around z_0 in a counter clockwise order.

With the given conditions $\pi_1(T^\circ_\lambda, z_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$ and the only relation among the γ_i is

$$\gamma_1\gamma_2\gamma_3\gamma_4 = \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4 = 1. \qquad (2)$$

Such a generating set is called a marking on T°_λ. It is not unique since any orientation preserving diffeomorphism of T°_λ takes a marking to a marking.

Exercise 9.2 Let δ be path from z_0 to z_1 and $\mathrm{Ad}_\delta : \alpha \to \delta\alpha\delta^{-1} = \delta * \alpha * \delta^{-1}$ the isomorphism $\pi_1(T^\circ_\lambda, z_0) \to \pi_1(T^\circ_\lambda, z_1)$, described in Exercise 9.1. Show that there is a marking T°_λ based at z_1 whose loops are homotopic to $\delta * \gamma_i * \delta^{-1}$. Thus we have a whole family of markings on T°_λ unique up to inner automorphism.

Since an epimorphism $\eta : \pi_1(T^\circ_\lambda, z_0) \to G$ is determined by the images of the generating set, the following "vector"

$$(a, b, c, d) = (\eta(\gamma_1), \eta(\gamma_2), \eta(\gamma_3), \eta(\gamma_4)) \in G^4$$

determines the epimorphism once a marking is selected. Now it turns out that since $q : S_\lambda \to T_\lambda$ has branching order k, l, m, n over w_1, w_2, w_3, w_4 respectively then

$$o(a) = k, o(b) = l, o(c) = m, o(d) = n \qquad (3)$$

Furthermore 9.2 implies that

$$abcd = 1. \qquad \text{(GV2)}$$

Finally since η is an epimorphism then

$$G = \langle a, b, c, d \rangle \,.$$

Notation 9.4 Such an (a, b, c, d) is called a *generating* (k, l, m, n)*-vector* or *quadruple*, analogous to generating triples for triangle tilings. The set of such vectors is denoted $GV((k, l, m, n), G)$. The group $\mathrm{Aut}(G)$ acts freely on $GV((k, l, m, n), G)$ we denote $GV((k, l, m, n), G)/\mathrm{Aut}(G)$ by $AGV((k, l, m, n), G)$.

We have the following proposition generalize our results for the tiling group of a triangle, and put conformal equivalence in terms of generating vectors.

Proposition 9.3 *Let $q : S_\lambda \to T_\lambda$ be the projection for a (k, l, m, n)-G-action on S_λ. Pick a specific $z_0 \in T^\circ_\lambda$ and a marking based at z_0 Then the G-action on S_λ determines a unique* $\mathrm{Aut}(G)$ *class of generating vectors. Correspondingly, given a generating (k, l, m, n)-vector a cover $q : S_\lambda \to T_\lambda$ is determined which is the quotient projection of a (k, l, m, n)-G-action.*

Proposition 9.4 *Let $q_1 : S_{\lambda_1} \to T_{\lambda_1}$ and $q_2 : S_{\lambda_2} \to T_{\lambda_2}$ be the projections of two (k, l, m, n)-G-action. Suppose that $h : T_{\lambda_1} \to T_{\lambda_2}$ is a conformal map preserving the order of branch points. Let (a, b, c, d) be the generating vector for q_1, computed with respect to a marking $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ on $T^\circ_{\lambda_1}$. Let $(\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4)$ the h_* image of*

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ on $T^\circ_{\lambda_2}$ and (a', b', c', d') be the generating vector for q_2 determined by $(\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4)$. Then the dotted arrow in the diagram

$$\begin{array}{ccc} S_{\lambda_1} & \overset{\widetilde{h}}{\dashrightarrow} & S_{\lambda_2} \\ q_1 \downarrow & & \downarrow q_2 \\ T_{\lambda_1} & \overset{h}{\longrightarrow} & T_{\lambda_2} \end{array}$$

may be filled in with a conformal map if and only if there is an $\omega \in \mathrm{Aut}(G)$ such that $(a', b', c', d') = \omega(a, b, c, d)$.In particular, if S_{λ_1} and S_{λ_1} are conformally equivalent by $\widetilde{h}$ and the G-action on S_{λ_1} is the $\widetilde{h}G\widetilde{h}^{-1}$ action, then h automatically exists.

Next we consider the case of conformal equivalence over the same quotient. In order to do this we need determine when T°_λ has nontrivial conformal equivalences. Since a conformal equivalence will induce a permutation of $\{0, 1, \infty, \lambda\}$ the possibilities will be restricted by the branching data symmetries and the value of λ. These results are given in Table 9.2. For the branching data (k, l, m, n), Let $\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ denote the conformal automorphism of $\mathbb{S}^2$ preserving the branch set $\{0, 1, \infty, \lambda\}$ as a set and preserving the branch orders (k, l, m, n). The group $\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ acts on the set of $\mathrm{Aut}(G)$-classes of epimorphisms $\pi_1(T^\circ_\lambda, z_0) \to G$ via:

$$(h, \eta) \to \eta \circ h_*^{-1} \circ \mathrm{Ad}_\delta$$

where δ is a path from z_0 to $h(z_0)$ and $\mathrm{Ad}_\delta : \pi_1(T^\circ_\lambda, z_0) \to \pi_1(T^\circ_\lambda, h(z_0))$ is the automorphism described in Exercise 9.1. If another path δ_1 is chosen then it is easily shown

$$\eta \circ h_*^{-1} \circ \mathrm{Ad}_{\delta_1} = \mathrm{Ad}_{\eta(h_*^{-1}(\delta_1 * \delta^{-1}))} \circ A\eta \circ h_*^{-1} \circ \mathrm{Ad}_\delta$$

leading to the same $\mathrm{Aut}(G)$-classes of epimorphisms. The action on generating vectors is calculated in the following way. Let (a, b, c, d) be a generating vector for η compute with respect to the marking $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Let $(\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4)$ be the $h_*^{-1} \circ \mathrm{Ad}_\delta$ image of the marking written as words in $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. We then substitute a, b, c, d in for $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ to calculate (a', b', c', d').

Example 9.1 Suppose that the branching data is $(2, 2, 3, 6)$, $\pi = (1, 2)$, and $\lambda = \frac{1}{2}$. Then the non-trivial of $\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ is $h(z) = 1 - z$. Pick $z_0 = \frac{1}{2} + i$ so that $h(z_0) = \frac{1}{2} - i$. Let γ_1 and γ_2 be loops created by straight lines from z_0 to counterclockwise loops about 0 and 1 respectively. The loop γ_3 travel vertically along $\mathrm{Re}(z) = \frac{1}{2}$ to meet a large clockwise circle centered at $\frac{1}{2}$. The loop γ_4 emanates from z_0 between γ_4 and γ_1 and moves around 0 to meet a small counterclockwise circle centered at $\frac{1}{2}$. Pick δ to travel from $\frac{1}{2} + i$ to $\frac{1}{4}$ to$\frac{1}{2} + i$. We get

$$\begin{aligned} \gamma'_1 &= \gamma_1^{-1}\gamma_4\gamma_1\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_1, \\ \gamma'_2 &= \gamma_1, \\ \gamma'_3 &= \gamma_1^{-1}\gamma_3\gamma_1, \\ \gamma'_4 &= .\gamma_1^{-1}\gamma_3^{-1}\gamma_1^{-1}\gamma_4\gamma_1\gamma_3\gamma_1 \end{aligned}$$

As a quick check we multiply out

$$\begin{aligned}\gamma_1'\gamma_2'\gamma_3'\gamma_4' &= \gamma_1^{-1}\gamma_4\gamma_1\gamma_2\gamma_1^{-1}\gamma_4^{-1}\gamma_1\gamma_1\gamma_1^{-1}\gamma_3\gamma_1\gamma_1^{-1}\gamma_3^{-1}\gamma_1^{-1}\gamma_4\gamma_1\gamma_3\gamma_1\\ &= \gamma_1^{-1}\gamma_4\gamma_1\gamma_2\gamma_3\gamma_1\\ &= \gamma_1^{-1}\gamma_4\gamma_1\gamma_2\gamma_3\gamma_4\gamma_4^{-1}\gamma_1\\ &= \gamma_1^{-1}\gamma_4\gamma_4^{-1}\gamma_1 = 1.\end{aligned}$$

Thus we get the transform

$$(a,b,c,d) \to (a',b',c',d') = (a^{-1}daba^{-1}d^{-1}a, a, a^{-1}ca, a^{-1}c^{-1}a^{-1}daca)$$

Notation 9.5 Even though there may not be a proper action of $\mathrm{Aut}(G)\times\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ on $GV((k,l,m,n),G)$, we call the quotient of the $AGV((k,l,m,n),G)$ classes of the generating vectors by $\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ the $\mathrm{Aut}(G)\times\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ classes of generating (k,l,m,n)-vectors of G and denote it by $CAGV((k,l,m,n),G,\lambda)$.

Proposition 9.5 *The surfaces S_λ with a (k,l,m,n)-G-action and lie over T_λ are in 1-1 correspondence with the* $\mathrm{Aut}(G)\times Aut_{(k,l,m,n)}(T_\lambda)$ *classes of generating (k,l,m,n)-vectors.*

Proposition 9.6 *Except in the cases (k,k,l,l) and (k,k,k,k), the $Aut_{(k,l,m,n)}(T_\lambda)$ action is trivial for generic λ.*

9.3 Lifting from moduli of quotients to moduli of surfaces

Let us first consider the problem of describing the family $S_\lambda \to T_\lambda$ of surfaces with (k,l,m,n)-G-action asT_λ varies through the family of ordered quotients. Here is what we have already. For each $\lambda \in \mathbb{S}^2 - \{0,1,\infty\}$ we have identified the conformal equivalence classes of surfaces $S_\lambda \to T_\lambda$ with a finite of set $CAGV((k,l,m,n),G,\lambda)$of $\mathrm{Aut}(G)\times\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ classes of generating (k,l,m,n)-vectors. Now consider the sets

$$\begin{aligned}\widetilde{\mathcal{M}}_{G,(k,l,m,n)} &= \{(v,\lambda) : v \in CAGV((k,l,m,n),G,\lambda), \lambda \in \mathbb{S}^2 - \{0,1,\infty\}\}\\ \widetilde{\mathcal{M}}_{(k,l,m,n)} &= \mathbb{S}^2 - \{0,1,\infty\}\end{aligned}$$

with the obvious map

$$q_1 : \widetilde{\mathcal{M}}_{G,(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}, (v,\lambda) \to \lambda.$$

This set captures all conformal equivalence classes of surfaces we are seeking, exactly once, except that some fibres are repeats of others. The goal of this section is to topologize $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$ that $\widetilde{\mathcal{M}}_{G,(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}$ is a branched cover of Riemann surfaces. The second goal of this section is to eliminate the redundant fibres by considering the Galois projection $q : \widetilde{\mathcal{M}}_{(k,l,m,n)} \to \mathcal{M}_{(k,l,m,n)} = (\mathbb{S}^2 - \{0,1,\infty\})/F$ as in Table 9.1 and finding $\mathcal{M}_{G,(k,l,m,n)}$ for which the diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{G,(k,l,m,n)} & \xrightarrow{\widetilde{p}} & \mathcal{M}_{G,(k,l,m,n)} \\ q_1 \downarrow & & \downarrow q_2 \\ \widetilde{\mathcal{M}}_{(k,l,m,n)} & \xrightarrow{p} & \mathcal{M}_{(k,l,m,n)} \end{array}$$

The object $\mathcal{M}_{G,(k,l,m,n)}$ is what we seek. We will work out the structure of $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$ as a warm up for determining $\mathcal{M}_{G,(k,l,m,n)}$.

Here is the scheme for topologizing $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$:

1. Identify the degenerate points $\mathcal{B}$ of $q_1 : \widetilde{\mathcal{M}}_{G,(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}$ The degenerate points will be those λ for which $\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ is higher than expected, and hence the fibre of q_1 could be smaller than expected. It is $0, 1$ or 2points and can be determined from the allowable λ column in Table 9.3.
2. Compute the monodromy action of $\pi_1(\widetilde{\mathcal{M}}_{(k,l,m,n)} - \mathcal{B})$ on a generic fibre, around punctures in $\widetilde{\mathcal{M}}_{(k,l,m,n)}$.
3. Determine the degeneration of fibres at the branch points by considering the action of the extra symmetries in $\mathrm{Aut}_{(k,l,m,n)}(T_\lambda)$ on the degenerate fibres. This will lead to multiple points in $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$.
4. Topologize $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$ via the Riemann Existence Theorem.
5. The orbits of the monodromy action on generic fibres determine the components of $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$ The genus and number of punctures for each component may be determined by the Riemann-Hurwitz formula.

The scheme for topologizing $\mathcal{M}_{G,(k,l,m,n)}$ is similar, except that we have :

2'.We need to look at monodromy over branch points of p.

3.'There should not be any multiple points introduced above the branch points.

Step 1 is easily calculated from the tables. Step 4 was discussed in Chapter 6, but here we need a simple modification of sewing in punctured discs over the punctures. Step 5 will follow from Step 2.

Ordered quotients Thus we need to work out procedure for Step 2 and Step 3. Before proceeding we need a construction to help us work with the parametrized family: $\lambda \to T_\lambda^\circ$.Let $\widetilde{\mathcal{M}}_{(k,l,m,n)} = \mathbb{S}^2 - \{0, 1, \infty\}$ as before and define the bundle of ordered quotients:

$$\widetilde{\mathcal{U}}^\circ_{(k,l,m,n)} = \{(w, \lambda) : w \in T_\lambda^\circ\} = \{(w, \lambda) \in \mathbb{S}^2 \times \mathbb{S}^2 : w \neq 0, 1, \infty, \lambda \neq 0, 1, \infty, w \neq \lambda\}.$$

The map $r : \widetilde{\mathcal{U}}^\circ_{(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}$ is a locally trivial fibre bundle and the fibre $r^{-1}(\lambda)$ is T_λ°. A short description of fibre bundles is given a the send of this section. We

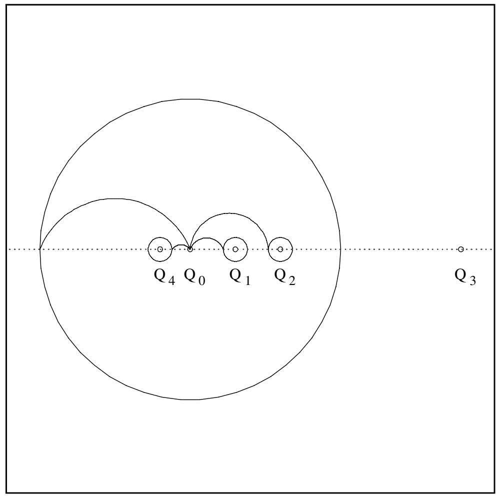

Figure 9.1: Figure 9.1 The δ's are missing.

will use the ideas of fibre bundles to construct the monodromy and to show how the variation in the modulus may be used to construct *a* parametrized family of surfaces, at least locally.

Fix a modulus λ_0 which is not a degenerate point for q_1 as excluded in Step Construct a marking $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ on $T^\circ_{\lambda_0}$ as usual. Next construct a marking $(\delta_1, \delta_2, \delta_3)$, based at λ_0, on $\widetilde{\mathcal{M}}_{(k,l,m,n)} = \mathbb{S}^2 - \{0, 1, \infty\}$,with these constraints.

- the paths $\delta_1, \delta_2, \delta_3$ emanate from λ_0 in a clockwise fashion,
- the paths $\delta_1, \delta_2, \delta_3$ encircle $0, 1, \infty$ in a clockwise fashion respectively,
- the path δ_i does not intersect γ_j if $i \neq j$ for $i, j \leq 3$.

Observe that even with the orientation changes $\delta_1\delta_2\delta_3 = 1$.

- See Figure 9.1

The constraints ensure that $\delta_1\delta_2\delta_3 = 1$.

Next pick a point lying over λ_0 namely a surface S_0 defined by the generating vector (a, b, c, d) and the marking $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Now consider moving λ_0 in the moduli space along the path δ_1. Since $r : \widetilde{\mathcal{U}}^\circ_{(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}$ is a locally trivial fibre bundle, there is a one parameter family of mappings

$$h_t : T^\circ_{\lambda_0} \to T^\circ_{\delta(t)}$$

with $h_0 = id$, and h_t fixing $0, 1$, and ∞. We may assume that the support of $h_t = \{x : h_t(x) \neq x\}$ is contained in a small neighbourhood of the path δ_1, one neighbourhood serving for all t. Correspondingly, the G-action defined on S_0 may be transported to a surface S_t lying over $T_{\delta(t)}$ by mapping the marking $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ to $(h_t(\gamma_1), h_t(\gamma_2), h_t(\gamma_3), h_t(\gamma_4))$ and then sending this marking to (a, b, c, d). Now consider filling in the dotted arrow labelled $\widetilde{h}_t$ in the following diagram.

$$\begin{array}{ccc} S_0^\circ & \overset{\widetilde{h}_t}{\dashrightarrow} & S_t^\circ \\ q_0 \downarrow & & \downarrow q_t \\ T_{\lambda_0}^\circ & \xrightarrow{h_t} & T_{\delta(t)}^\circ \end{array}$$

The lifting criterion

$$h_{t*} \circ q_{0*}(\pi_1(S_0^\circ)) = q_{t*}(\pi_1(S_t^\circ))$$

is satisfied since $q_{t*}(\pi_1(S_t^\circ))$ was manufactured precisely to satisfy this condition. Thus we have what appears to be a continuously varying family of surfaces lying over δ_1. The surfaces S_0 and S_1 both lie over T_{λ_0},let see what their relationships is. Define $(\gamma_1', \gamma_2', \gamma_3', \gamma_4') = (h_1(\gamma_1), h_1(\gamma_2), h_1(\gamma_3), h_1(\gamma_4))$ and η_0 defining S_0 and hence satisfying the epimorphism $\eta_0(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (a, b, c, d)$. Since S_1 is the surface defined by η_1 satisfying $\eta_1(\gamma_1', \gamma_2', \gamma_3', \gamma_4') = (a, b, c, d)$ then $\eta_1 \circ (h_1)_* = \eta_0$ or $\eta_1 = \eta_0 \circ (h_1^{-1})_*$. Now given a specific δ it is actually easier to calculate $\eta_0 \circ (h_1)_*$ by substitution as done in the last section. Here are the formulas for $(a, b, c, d) = \eta_0(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

δ	point encircled by δ	$(a', b', c', d') = \eta_0 \circ (h_1)_*(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$
δ_1	0	$(a^{-1}d^{-1}ada, b, c, a^{-1}da)$
δ_2	1	$(a, b^{-1}a^{-1}d^{-1}aba^{-1}dab, c, ab^{-1}a^{-1}daba^{-1})$
δ_3	∞	$(a, b, d^{-1}cd, d^{-1}c^{-1}dcd)$

We did not consider the degenerate for q_1 above. Indeed loops in $\widetilde{\mathcal{M}}_{(k,l,m,n)}$ encircling only degenerate point are homotopically trivial and so h_1 is homotopic to the identity and the monodromy action is trivial. So what happens? We may construct an unramified cover $\widetilde{\mathcal{M}}'_{G,(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}$ from the monodromy action This the total space of epimorphisms $\pi_1(T_\lambda^\circ) \to G$ up to $\mathrm{Aut}(G)$-equivalence. There is a map $\widetilde{\mathcal{M}}'_{G,(k,l,m,n)} \to \widetilde{\mathcal{M}}_{G,(k,l,m,n)}$ in which some of the points in the fibres over degenerate points are are identified. this results in an curve with singularities. In the parlance of algebraic geometry $\widetilde{\mathcal{M}}'_{G,(k,l,m,n)}$ is the normalization of $\widetilde{\mathcal{M}}_{G,(k,l,m,n)}$. The map $\widetilde{\mathcal{M}}'_{G,(k,l,m,n)} \to \widetilde{\mathcal{M}}_{G,(k,l,m,n)}$ is 1-1 except at a finite number of points lying over the degenerate points.

Unordered quotients First we are going to construct the bundle of unordered quotients $r : \mathcal{U}^\circ_{(k,l,m,n)} \to \mathcal{M}_{(k,l,m,n)}$. Let F denote the finite group of linear fractional transformations of determined by the allowable symmetries per Table 9.3. The group F acts on $\widetilde{\mathcal{U}}^\circ_{(k,l,m,n)}$ by

$$h \cdot (w, \lambda) \to (h(w), h(\lambda))$$

and the projection $r : \widetilde{\mathcal{U}}^\circ_{(k,l,m,n)} \to \widetilde{\mathcal{M}}_{(k,l,m,n)}$ satisfies $r(h \cdot (w, \lambda)) = h \cdot \lambda = h \cdot r(w, \lambda)$.

Let $\mathcal{U}^\circ_{(k,l,m,n)} = \widetilde{\mathcal{U}}^\circ_{(k,l,m,n)}/F$ then it follows the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{U}}^\circ_{(k,l,m,n)} & \xrightarrow{F} & \mathcal{U}^\circ_{(k,l,m,n)} \\ r\downarrow & & \downarrow \overline{r} \\ \widetilde{\mathcal{M}}_{(k,l,m,n)} & \xrightarrow{p} & \mathcal{M}_{(k,l,m,n)} \end{array}$$

commutes. The horizontal maps are group quotients and the specific format of p as a rational function is given in Table 9.1. Now let us identify the fibres of $\overline{r}$. If $\overline{\lambda} = p(\lambda)$ is not a branch point of p then $p^{-1}(\overline{\lambda}) = \bigcup_{g \in F} T^\circ_{g\lambda}/F$.We may think of this fibre as several different $T^\circ_{g\lambda}$ glued together by the F action. Now fix a λ_0 and pick a marking $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ on $T^\circ_{\lambda_0}$; the gluing translates this marking to $g_*(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Now suppose that S_{λ_0} is a surface with G-action lying over surface T_{λ_0} defined by the epimorphism $\eta_1 : \pi_1(T^\circ_{\lambda_0}) \to G$. Then, by the lifting criterion, the surface $S_{g\lambda_0}$ lying over $T_{g\lambda_0}$ via $\eta_1 \circ g_*^{-1}$ will be conformally equivalent to S_{λ_0}. Thus we should choose $\eta_g = \eta_1 \circ g_*^{-1}$ to get a compatible set of epimorphisms $\eta_g : \pi_1(T^\circ_{g\lambda_0}) \to G$, all leading to a conformally equivalent surface lying over the $T_{g\lambda}$.
Let $\mathcal{M}^\circ_{(k,l,m,n)}$ denote $\mathcal{M}_{(k,l,m,n)}$ minus the branch points of p. Consider a loop $\delta(t)$ in $\mathcal{M}^\circ_{(k,l,m,n)}$ at $\overline{\lambda_0} = F\lambda_0$,and $\lambda(t)$ its lift to$\widetilde{\mathcal{M}}_{(k,l,m,n)}$ starting at λ_0. Then there is a local trivialization $h_t : T^\circ_{\lambda_0} \to T^\circ_{\lambda(t)}$, and hence compatible set of local trivializations $gh_t : T^\circ_{g\lambda} \to T^\circ_{g\lambda(t)}$, and, further, a local trivialization of the glued fibres $\overline{h_t} : T^\circ_{\lambda_0}/F \to T^\circ_{\lambda(t)}/F$. At $t = 1$ we have $h_1(F\lambda_0)$ as a set but not necessarily $h_1(g\lambda_0) = g\lambda_0$. Now by compatibility and the lifting criterion we get a family of trivializations $\widetilde{gh_t} : S_{\lambda_0} \to S_{g\lambda(t)}$. The point in $\mathcal{M}_{G,(k,l,m,n)}$ we are looking for, produced by the monodromy action of δ is any of the surfaces $S_{g\lambda(1)}$. Now how do we specifically compute $S_{g\lambda(1)}$ in terms of the generating vectors at. λ. The local trivializations produce transformed markings in the quotients $T^\circ_{g\lambda(1)}$, $g \in F$.To compare the original and the monodromic image we need to compare the original marking in to the marking in $T^\circ_{g\lambda(1)}$ where $g\lambda(1) = \lambda_0$. Thus the we need to perform three operations:

1. Compute the image $(\gamma_1'', \gamma_2'', \gamma_3'', \gamma_4'')$ of the marking $T^\circ_{\lambda(1)}$
2. Map the marking back to T°_λ by the map g, i.e., find $(\gamma_1''', \gamma_2''', \gamma_3''', \gamma_4''') = g(\gamma_1'', \gamma_2'', \gamma_3'', \gamma_4'')$
3. Adjust the base point of the marking to get $(\gamma_1', \gamma_2', \gamma_3', \gamma_4') = \mathrm{Ad}_\zeta(\gamma_1''', \gamma_2''', \gamma_3''', \gamma_4''')$ for some appropriately chosen ζ connecting the bases points of the markings.

The loops δ, based at λ_0 should be be chosen so that

- the paths $\delta_1, \ldots, \delta_t$ emanate from λ_0 in a clockwise fashion,
- the paths $\delta_1, \ldots, \delta_t$ encircle either a branch point or a puncture in $\mathcal{M}_{(k,l,m,n)}$,
- the path δ_i does not intersect γ_j unless δ_i encircles the puncture encircled by γ_j.

Remark 9.2 It turns out that $t = 3$ in all cases. If δ_i does not encircle a branch point or a puncture which is also a branch point, then the transformation in part 2 is trivial. The transform in part 1 will only affect those γ_j that meet δ_i.

Here is the carrt of transforms for branching data type (k, k, l, m)

δ	point encircled by δ	$(a', b', c', d') = \eta_0 \circ (h_1)_*(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$
δ_1	0	$(a^{-1}d^{-1}ada, b, c, a^{-1}da)$
δ_2	1/2	$(b^{-1}a^{-1}b, b^{-1}a^{-1}bab, c, d)$
δ_3	∞	$(a, b, d^{-1}cd, d^{-1}c^{-1}dcd)$

9.4 Moduli of quadrilateral tilings

How is the modulus λ related to tilings?

Proposition 9.7 *Let the notation is as above, and that S_λ is a surface with (k, l, m, n)-action by G. and that the map $S_\lambda \to S_\lambda/G$ is branched over $0, 1, \infty$, and λ, with branching orders k, l, m, and n respectively. Then the G-action arises from a tiling by quadrilaterals only if λ is real and*

- if $\lambda < 0$ then the quadrilaterals are (k, l, m, n)-quadrilaterals,
- if $0 < \lambda < 1$ then the quadrilaterals are (k, n, l, m)-quadrilaterals,
- if $\lambda > 1$ then the quadrilaterals are (k, l, n, m)-quadrilaterals.

Proof. If S_λ is tiled by quadrilaterals then reflection induced on the Riemann sphere must be complex conjugation since 0,1, and ∞ are fixed. Thus $\lambda = \overline{\lambda}$ and λ must be real. The rest of the proposition is easily demonstrated.

Remark 9.3 For polygons with more sides we may take the moduli space of surfaces with G-action to be $\{(\lambda_4, \lambda_5, \ldots, \lambda_n) : \lambda_i \in \mathbb{C} - \{0, 1\}, \lambda_i \neq \lambda_j\}$. If any of the branching orders are equal we need to divide by the action of a suitable finite group. The space of polygon tilings is $\{(\lambda_4, \lambda_5, \ldots, \lambda_n) : \lambda_i \in \mathbb{R} - \{0, 1\}, \lambda_i \neq \lambda_j\}$. Thus for pentagons the moduli space is $\mathbb{R}^2 - (\{\lambda_4 = 0, 1\} \cup \{\lambda_4 = 0, 1\} \cup \{\lambda_4 = \lambda_5\})$, consisting of 12 regions in the plane.

Remark 9.4 Another possibility for a symmetry is for $k = l$ and for ϕ to interchange w_1 and w_2 and to fix w_3 and w_4. In this case the corresponding surfaces will form a different component of $\mathcal{M}_\sigma(\mathbb{R})$.

9.5 Addendum: Permutations to LFT's

It turns out that there is a finite group of transformations F, and a map $\ell : \Sigma_4 \to F$ such that for $\pi \in \Sigma_4$ and the corresponding ℓ_π that

$$\chi(w_{\pi(1)}, w_{\pi(2)}, w_{\pi(3)}, w_{\pi(4)}) = \ell_\pi(\chi(w_1, w_2, w_3, w_4))$$

If we write $K_4 = \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ and $\Sigma_3 = \langle(1,2),(1,2,3)\rangle$ Then $\Sigma_4 = \Sigma_3 \ltimes K_4$ and ℓ is trivial on K_4.The results are summarized in this table:

Table 9.2 - Transforms of the space of ordered quotients.

$\pi' \in \Sigma_3$	$\pi \in \pi' K_4 \subseteq \Sigma_4$	$\ell_\pi(\lambda)$	fixed points
1	$\{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$	λ	all
$(1,2)$	$\{(1,2), (3,4), (1,4,2,3), (1,3,2,4)\}$	$1-\lambda$	$\frac{1}{2}, \infty$
$(1,3)$	$\{(1,3), (1,4,3,2), (2,4), (1,2,3,4)\}$	$1/\lambda$	± 1
$(2,3)$	$\{(2,3), (1,2,4,3), (1,3,4,2), (1,4)\}$	$\lambda/(\lambda-1)$	$0, 2$
$(1,2,3)$	$\{(1,2,3), (2,4,3), (1,4,2), (1,3,4)\}$	$(\lambda-1)/\lambda$	$\frac{-1}{2} \pm \frac{\sqrt{3}}{2} i,$
$(1,3,2)$	$\{(1,3,2), (1,4,3), (2,3,4), (1,2,4)\}$	$1/(1-\lambda)$	$\frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$

In the table to follow we present all conformal automorphisms of T_λ that induce the permutation $(w_1, w_2, w_3, w_4) \to (w_{\pi(1)}, w_{\pi(2)}, w_{\pi(3)}, w_{\pi(4)})$. For $\pi = \pi_1\pi_2$, with $\pi_1 \in \Sigma_3$ and $\pi_2 \in K_4$ a transformation can be found for each λ if the transformation $\pi_1 = 1$ and for λ in the fixed point set of $\ell_{\pi_1}(\lambda)$ but not in $\{0, 1, \infty\}$. These fixed points are listed in Table 9.2. To find the transformation for a general π one need only need to use the homomorphism property $\Sigma_4 \to PSL_2(\mathbb{C})$.

Table 9.3- Conformal automorphisms of T_λ°.

$\pi = \pi_1\pi_2$	allowable λ	LFT	fixed points
1	all	$z \to z$	all
$(1,2)(3,4)$	all	$z \to \lambda(z-1)/(z-\lambda)$	$\lambda \pm \sqrt{\lambda^2 - \lambda}$
$(1,3)(2,4)$	all	$z \to \lambda/z$	$\pm\sqrt{\lambda}$
$(1,4)(2,3)$	all	$z \to (z-\lambda)/(z-1)$	$1 \pm \sqrt{1-\lambda}$
$(1,2)$	$\frac{1}{2}$	$z \to 1-z$	$\frac{1}{2}, \infty$
$(1,3)$	-1	$z \to 1/z$	$1, -1$
$(2,3)$	2	$z \to 1/(1-z)$	$0, 2$
$(1,2,3)$	$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$	$z \to 1/(1-z)$	$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
$(1,3,2)$	$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$	$z \to (z-1)/z$	$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Remark 9.5 .

The moduli space is not usually compact. Indeed in the case of (k, l, m, n) G-actions. The moduli space is a compact sphere minus three points. The moduli space can be compactified by adding certain surface with singular points, typically double points (take two points on a smooth surface and pinch them together. An interesting problem cited below is to describe how the G-symmetry extends to these singular surfaces.

9.6 Addendum: Locally trivial families fibre bundles.

Definition 9.6 A map $p : E \to B$ is a (differentiably) locally trivial fibre bundle if for each $y \in B$ there is a neigbourhood U of y and a trivializing homeomorphism (diffeomorphism) $h : p^{-1}(y) \times U \to p^{-1}(U)$ such that $p(h(x, y)) = y$. The space B and E are called the base space and the total space respectively and $p^{-1}(y)$ is called the fibre above y.

Example 9.2 Let $E = \mathbb{R}^3 - \{0\}$, $B = (0, \infty) = \mathbb{R}^+$and $p : E \to B$ the map $x \to x/\|x\|$. Thus $p^{-1}(r) = S_r$ is the sphere of radius r. If we pick $r_0 = 1$ then the trivializing homeomorphism actually works for all of E, namely $h : S_1 \times \mathbb{R}^+ \to E$, $h(x, r) = rx$. This is just three dimensional polar coordinates.

Example 9.3 A covering space is a locally trivial bundle with discrete fibres.

For a locally trivial fibre bundle and a specific $y_0 \in U$ the map $h_y : p^{-1}(y0) \to p^{-1}(y)$ given by $h_y(x) = h(x, y)$ is a homeomorphism (diffeomorphism), i.e., $y \to h_y$ gives us a parametrized family of homeomorphisms of the distinguished fibre to a general fibre. Thus we describe the parametrized family $y \to p^{-1}(y)$ as a locally trivial family of spaces. If the base space is connected then any two fibres are homeomorphic(diffeomorphism). Thus fibre bundle notions are very useful for classification problems since we need only show that two different spaces are fibres of a bundle with a connected base space. The following theorem allows fibre bundles to be used extensively:

Theorem 9.8 *Ehresmann Fibration Lemma. Let $p : E \to B$ be a proper,submersive map of manifolds, i.e., $p^{-1}(K)$ is compact for every compact $K \subseteq B$ and the differential $dp_x : T_x(E) \to T_{p(x)}(B)$ is onto for every $x \in E$. Then $p : E \to B$ is a differentiably) locally trivial fibre bundle.*

Later the following extension was proven to deal with locally trivial families with non compact fibres.

Theorem 9.9 *Ehresmann Fibration Lemma. Let $p : E \to B$ be a proper,submersive map of manifolds, and assume that $E_1 \subseteq$ is a closed submanifold such that p restricted to E_1 is a proper submersion. Then a local trivialization of $p : E \to B$ can be found that restricts to trivialization of $p : E_1 \to B$. Consequently $p : E \backslash E_1 \to B$ is locally trivial.*

Here the main tool to help us deal with locally trivial families that are not globally trivial:

Proposition 9.10 *Let $p : E \to B$ be differentiably locally trivial and let $\gamma \in \pi_1(B, y_0)$ then there is a parametrized family of homeomorphisms $h_{\gamma,t} : p^{-1}(y0) \leftrightarrow p^{-1}(\gamma(t))$. The $h_{\gamma,t}$ is well defined up to homotopy. Thus there is a map $\pi_1(B, y_0) \to \mathrm{Diff}(p^{-1}(y0))$, called the monodromy map, given by $\gamma \to h^{-1}_{\gamma,1}$.The homeomorphism $h_{\gamma,1}$ is only well defined up to isotopy. For a covering space we get the monodromy map as previously discussed.*

9.7 REU Problems

Problem R9.1 Determine all the moduli spaces of $G-(k,l,m,n)$ actions of surfaces of given low genus.

Problem R9.2 Is $\mathcal{M}_{G,(k,l,m,n)}$ smooth?

Problem R9.3 Does there exist a universal family $r : U_{G,(k,l,m,n)} \to M_{G,(k,l,m,n)}$ such that $r^{-1}(w)$ is a curve whose modulus is w.

Problem R9.4 Determine the subsets of the moduli space corresponding to tilings.

Problem R9.5 Determine all the divisible tilings as special subsets of the moduli space of tilings. I.e., for what values of the moduli are the tilings subdivided by a smaller tiling. In the case of quadrilaterals for instance special values of λ will allow for a subdivision. See [8]

Problem R9.6 As $\lambda \to 0$, $\lambda \to 1$, or $\lambda \to \infty$, describe the degeneration of the quadrilateral tiling on a surface S. In particular it would be interesting to describe the tiling on the limiting singular surface. Describe the G-action on the limiting singular surfaces. Work with four point branch surfaces first.

Problem R9.7 Find all the equisymmetric strata of the moduli space of surfaces of low genus.