Quadrilaterals Subdivided by Triangles in the Hyperbolic Plane

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Abstract

In this paper, we consider triangle-quadrilateral pairs in the hyperbolic plane which "kaleidoscopically" tile the plane simultaneously. These tilings are called *divisible tilings* or *subdivided tilings.* We restrict our attention to the simplest case of divisible tilings, satisfying the *corner condition*, in which a single triangle occurs at each vertex of the quadrilateral. All possible such divisible tilings are catalogued as well as determining the minimal genus surface on which the divisible tiling exists. The tiling groups of these surfaces are also determined.

1 Introduction

Let S be a surface which is a two-dimensional object such as the covering of a sphere or torus. Every surface may be represented as a sphere with σ handles where σ is the genus of the surface. For example, a sphere (Figure 1) has genus 0 while a torus (Figure 2) has genus 1. A genus 2 surface is pictured in Figure 3. A surface can be *tiled* if it can be completely covered by a series of non-overlapping polygons, called *tiles,* as in Figures 1 and 2. A similar definition of tilings also applies for the euclidean or hyperbolic plane.

Note in the tiling of the torus in Figure 2, that two tiles meeting along a long edge may be clumped together to form a quadrilateral, and that the quadrilaterals so formed tile the surface. There are several other ways of clumping the tiles together to form quadrilaterals that tile: at least two ways with four triangles and at least three ways with eight triangles. We say that the quadrilateral tiling is *subdivided* by the triangle tiling or that

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the quadrilateral tiling is *divisible*. In this report, we investigate which tilings of surfaces by quadrilaterals may be subdivided by triangle tilings. A complete solution is beyond the scope of this report so we make the following restrictions:

1. The tiling is "kaleidoscopically symmetric", as described in the next section, since the problem is then tractable and interesting.
2. The genus of S satisfies $\sigma \geq 2$, the sphere and torus examples are well-known.
3. There is only one triangle in each corner of the quadrilateral. Note that this does not hold for all our examples in Figure 2. This is called the *corner condition*.

The investigation will proceed in two stages, first we find the divisible quadrilateral tilings in the hyperbolic plane $\mathbb{H}$ (universal cover) and then secondly determine the minimal surfaces that can be covered by such tilings. It turns out that the corner condition guarantees that there is a unique minimal surface which simultaneously supports both tilings.

The main results of the paper are a catalogue of the divisible quadrilateral tilings satisfying the corner condition (Theorem 13), the determination of the minimal surface on which the two tilings can be simultaneously drawn (Theorem 18), and a preliminary investigation of the corresponding tiling groups (Theorem 18). A tabulation of the tiling, genus and group information in the 13 cases discovered is given in Table 1 of section 8. Pictures of the various tiled quadrilaterals are given in Table 2 of the same section.

The remainder of the report is structured as follows. In section 2 we introduce the necessary background on tilings on surfaces and in the plane. In sections 3 and 4 we introduce the algorithm for determining which tilings are divisible and work out some sample calculations. In section 5 a discovered symmetry pattern of the tilings of the quadrilaterals is discussed. The minimal genus of a surface exhibiting the divisible tiling is discussed in section 6. Finally a list of further research questions is presented in section 7, followed by the tables of results in section 8. Throughout the discussion the reader is advised to look at the pictures of divisible tilings in section 8 to gain a clearer idea of the definitions and discussion.

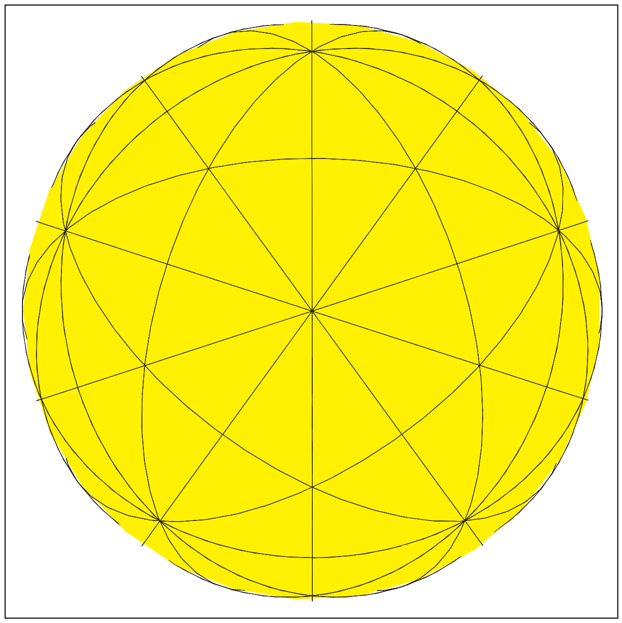

Figure1. Icosahedral tiling of the sphere

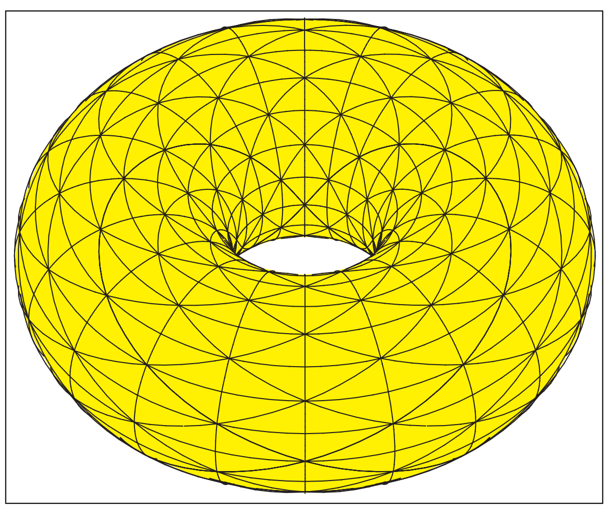

Figure 2. 2-4-4 tiling on a torus

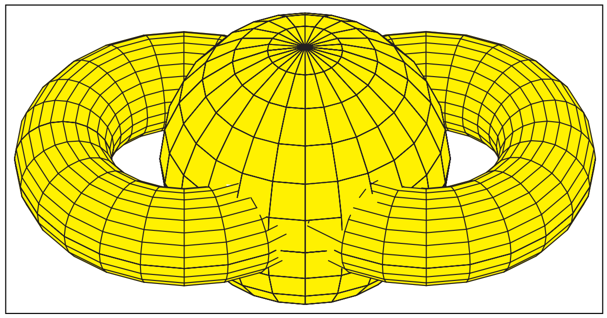

Figure 3. A genus 2 surface - sphere with two handles

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2 Introduction to tilings

Definition 1 *A tiling of a surface is said to be a* geodesic, kaleidoscopic tiling *if the following two conditions are met:*

1. *For each edge e of the tiling there is a mirror reflection r_e of the entire surface, fixing all the points of e, and interchanging the two polygons that meet along e.*

2. *The* mirror $\{x \in S : r_e(x) = x\}$ *of each reflection r_e is the union of edges of the tiling. The components of the mirror are called* ovals.

A *geodesic, kaleidoscopic tiling* of the euclidean or hyperbolic plane has a similar definition except that the mirror of the reflection is a line which is a union of edges. We shall assume that all our tilings satisfy these conditions.

A tiling is said to be *divisible* if it can be divided into a finer tiling. We have seen an example of a divisible tiling in Figure 2 and many examples of divisible tilings of the hyperbolic plane are given in section 8. There we only show one quadrilateral, though it may be easily extended to the plane through reflections in the sides of the quadrilateral. As the examples show, we typically have pair of a triangle Δ and quadrilateral Ω, with $\Delta \subset \Omega$, and Ω is a union of triangles of the tiling defined by Δ.

Remark 2 *The surface or plane has a metric in which the reflections r_e are isometries. It follows that each oval in the mirror on a surface is a simple, closed, geodesic curve. On the sphere the metric is the natural one derived from its embedding in 3-space. Higher genus surfaces must be embedded in a higher dimensional space to realize the metric.*

The hyperbolic plane As previously mentioned, we will first solve the problem by finding divisible tilings in the universal cover of the surface. Since $\sigma \geq 2$, we will therefore be using hyperbolic geometry throughout the report. We have chosen to use the disc model for the hyperbolic plane $\mathbb{H}$, in which the points are in the interior of the unit disc, the lines are the unit disc portions of circles and lines perpendicular to the boundary of the unit disc, and reflections are inversions in the circles defining the lines. All properties we use about hyperbolic geometry, such as area formulas for polygons, may be found in the text by Beardon [1]. It is important to note that in hyperbolic geometry, the angle sum of triangles is less than 180 degrees. Also note that for figures drawn on the hyperbolic plane, as in section 8, triangles on the outer edge of the disc appear to be smaller than those in the center. However, all triangles of these are congruent in hyperbolic geometry.

2.1 The master tile and the tiling group

The entire kaleidoscopic, geodesic tiling of a surface or the plane can be generated by a single tile, called the *master tile,* pictured in Figure 4. The sides of the master tile, Δ_0, are labeled p, q, and r, and we denote the points opposite these sides by P, Q, and R, respectively. We also denote by p, q, and r the reflection in corresponding side. As a consequence of the geodesic condition each vertex has a even number of triangles around it, as in Figure 1. Hence the angles at P, Q, R have size $\frac{\pi}{l}$ radians, $\frac{\pi}{m}$ radians, and $\frac{\pi}{n}$ radians, respectively (as shown in Figure 4) where l, m, and n are integers ≥ 2. We call such a triangle an (l, m, n)-triangle. For example, the icosahedral tiling in Figure 1 is a $(2, 3, 5)$-tiling.

At each of the vertices of the triangle, the product of the two reflections in the sides of the triangle meeting at the vertex is a rotation fixing the vertex. The angle of rotation is twice the angle at this vertex. For example the product pq - a reflection first through q then through p is a counter-clockwise rotation through $\frac{2\pi}{l}$ radians. We will refer to this rotation as $a = pq$ and also use it to label the vertex in Figure 4. Rotations around each of the other corners can be defined in the same way, so that $b = qr$ is a counterclockwise rotation through $\frac{2\pi}{m}$ radians and $c = rp$ is a counterclockwise rotation through $\frac{2\pi}{n}$ radians.

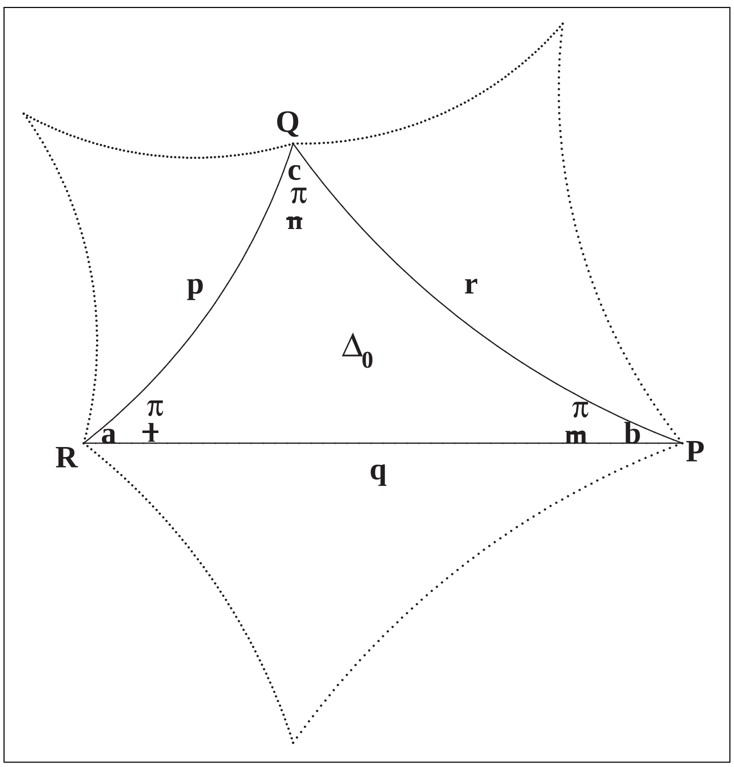


Figure 4. The master tile Δ_0, reflected images, and generators of T^* and T

From the geometry of the master tile, we can derive relations among these group elements. It is clear that since p, q, and r are reflections, the order of each of these elements is 2:

$$p^2 = q^2 = r^2 = 1. \tag{1}$$

From the observations about rotations above, it is also clear that

$$a^l = b^m = c^n = 1, \tag{2}$$

and

$$abc = pqqrrp = 1. \tag{3}$$

The reflections generate a group

$$G^* = \langle p, q, r \rangle ,$$

and the rotations generate a subgroup $G = \langle a, b, c \rangle$ which includes only the orientation-preserving isometries in G^*. Note that $G = \langle a, b \rangle$ since $c = (ab)^{-1}$. The subgroup G is of index 2 in G^* and hence is also normal in G^*.

The genus of the surface is related to the group order and the (l, m, n)-triple by the Riemann-Hurwitz equation:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right) = \mu(l, m, n). \tag{4}$$

A table of values of $\mu(l, m, n) \leq \frac{1}{4}$ for hyperbolic triangles is included in Appendix A.

Remark 3 *For a tiling in the hyperbolic plane the relations above are the only relations. The groups are now infinite and we denote them by T^* and T.*

$$T^* = \left\langle p, q, r : p^2 = q^2 = r^2 = (pq)^l = (qr)^m = (rp)^n = 1 \right\rangle . \tag{5}$$

$$T = \left\langle a, b, c : a^l = b^m = c^n = abc = 1 \right\rangle .$$

Remark 4 *The groups G^* and T^* are related by $G^* \backsimeq T^*/\Gamma$ for a normal subgroup Γ of hyperbolic translations of $\mathbb{H}$. This is discussed at greater length in section 6.*

Now, q acts by conjugation on the generators of G, inducing an automorphism θ which maps the generators to their inverses:

$$\theta(a) = qaq^{-1} = q(pq)q = qp = q^{-1}p^{-1} = (pq)^{-1} = a^{-1} \tag{6}$$

and

$$\theta(b) = qbq^{-1} = q(qr)q = rq = r^{-1}q^{-1} = (qr)^{-1} = b^{-1}. \qquad (7)$$

This makes $G^* = \langle q \rangle \ltimes_\theta G$, where θ is an *involutary automorphism* since $q^2 = 1$.

Remark 5 *Similar remarks apply for a quadrilateral that kaleidoscopically tiles except that the angles of the quadrilateral are* $\frac{\pi}{s}$, $\frac{\pi}{t}$, $\frac{\pi}{u}$, *and* $\frac{\pi}{v}$ *for some integers,* s, t, u, *and* v, *and* G^* *and* G *have 4 generators instead of three. The relations are similar.*

The above discussion on θ and quadrilaterals also applies to T^* and T.

2.2 Divisible tilings

As previously noted, we are focusing on triangles which subdivide quadrilaterals, say defined by the pair $\Delta \subset \Omega$. The area of the triangle, A_t, is given by the following formula (see [1])

$$A_t = \pi\mu(l, m, n) = \pi\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right).$$

Since the quadrilateral also tiles the plane it has angles $\frac{\pi}{s}$, $\frac{\pi}{t}$, $\frac{\pi}{u}$, and $\frac{\pi}{v}$ for some integers, s, t, u, and v, as noted in Remark 5 above. The area of the quadrilateral is given by the following:

$$A_q = \pi\mu(s, t, u, v) = \pi\left(2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}\right).$$

Note that we must therefore have

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1 \text{ and } \frac{1}{s} + \frac{1}{t} + \frac{1}{u} + \frac{1}{v} < 2.$$

Let K denote the number of triangles which divide the quadrilateral. Because the triangles are all congruent we have the following relationship:

$$2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v} = K\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right), \qquad (8)$$

or alternatively:

$$K = \frac{2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}}{1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}}. \qquad (9)$$

Remark 6 *For a triangle to tile a quadrilateral, each of s, t, u, and v, must be a divisor of one of l, m, or n, since some multiple of a triangle angle must 'fit' into each corner of the quadrilateral. However, since we are only looking at quadrilaterals that satisfy the corner condition then each s, t, u, and v must equal one of l, m, or n.*

3 An algorithm for determining divisible tilings

Given a specific (s, t, u, v) quadrilateral Ω and (l, m, n) triangle Δ, satisfying the area equation 8, our algorithm determines whether or not there exists a kaleidoscopic tiling of the plane by Ω which is subdivided by a kaleidoscopic tiling by Δ. Our algorithm starts with a kaleidoscopic tiling by triangles on the hyperbolic plane and determines if a quadrilateral can be drawn along the lines of the triangular tiling. Since the sides of the quadrilateral are formed from the sides of the triangles then the reflection property is automatically satisfied. Since the size of the angles of Ω are integer submultiples of π the geodesic property of the quadrilateral tiling is also guaranteed, and the quadrilaterals do not overlap. Before discussing our algorithm we need to introduce a few ideas which considerably restrict the number of quadrilateral searches to be done. In section 4 we describe how to organize the search and illustrate the various components of the algorithm through sample calculations.

3.1 The number of triangles

The number, K, of triangles contained in each quadrilateral, is given by 9. Thus the right hand side of 9 must be an integer. It also turns out that K has an upper bound of 60 for all triangles in the hyperbolic plane.

Proposition 7 *Suppose the (l, m, n)-triangle Δ tiles the (s, t, u, v)-quadrilateral Ω. We do not assume that the corner condition is satisfied. Then*

$$K = \frac{2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}}{1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}} \leq 60.$$

Proof. Fix l, m and n. To maximize K we need to maximize the area of a quadrilateral. Thus each integer s, t, u, and v, should be made as large as possible. Since each of s, t, u, and v must divide one of l, m, or n, then the largest possible quadrilateral is a (b, b, b, b)-quadrilateral such that b is the largest integer selected from l, m, and n. Now the smallest possible value of $\mu = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$ on the hyperbolic plane is $\frac{1}{42}$ for a $(2,3,7)$-triangle. Picking a $(7,7,7,7)$-quadrilateral as suggested above, we get $K = \frac{10}{7}/\frac{1}{42} = 60$. For any other triangle we have $\mu \geq \frac{1}{24}$ with $\mu = \frac{1}{24}$ realized for a $(2,3,8)$ triangle. But now

$$K = \frac{2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}}{1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}} \leq \frac{2}{1/24} = 48.$$

Thus 60 is the maximum possible value of K. ■

From the other end we have:

Proposition 8 *Suppose the (l,m,n) triangle Δ tiles the (s,t,u,v)-quadrilateral Ω, such that the corner condition is satisfied. Then*

$$K = \frac{2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}}{1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}} \geq 6.$$

Proof. Each of the four vertices of the quadrilateral corresponds to a distinct triangle. Otherwise, the corner condition is violated or the quadrilateral is really a triangle. Now each of these four corner triangles has exactly two sides on the boundary of the quadrilateral, and one side interior to the quadrilateral. The corner triangles cannot meet along the interior edges, for again, the corner condition is violated or the quadrilateral is really a triangle. There must be at least two additional triangles to match these four internal edges. The catalogue shows that there is an exactly one example of a divisible tiling satisfying the corner condition with $K = 6$. ■

3.2 Hubs

An *l-hub* is a collection of l (l,m,n)-triangles arranged such that the $\frac{\pi}{l}$ radian angles meet at a single vertex, forming a straight angle of π radians along an oval or geodesic. An m-hub and n-hub are defined in the same fashion. When examining a subdivided quadrilateral, most of the hubs will appear on the boundary of the quadrilateral, as illustrated in the figures in section 8. However, there are also hubs which appear in the interior of the quadrilateral as two adjacent hubs, one on each side of an oval. The collection of these $2l$ triangles is called a *double l-hub* or an *interior hub.*

The number of l-hubs, h_l, in each subdivision can be determined by the following formula:

$$h_l = \frac{Kx_l - c_l}{l}, \tag{10}$$

where x_l is the number of $\frac{\pi}{l}$-angles in the triangle and c_l is the number of quadrilateral corners of size $\frac{\pi}{l}$. We can clearly see that h_l must be an integer simply by the definition of l-hub. The only instance in which a partial l-hub could appear would be in the vertex of a quadrilateral. However, this case is easily ruled out because we are only examining quadrilaterals with single triangles at the vertices. Similar formulas hold for m-hubs and n-hubs. There are always hubs of every type, as we now prove.

Proposition 9 *For all quadrilaterals subdivided by an (l, m, n)-triangle, and satisfying the corner condition, there exists at least one l-hub, one m-hub, and one n-hub.*

Proof.
Note that

$$h_i = \frac{Kx_i - c_i}{i}, \tag{11}$$

where $i = l, m$, or n. In order for $h_i = 0$, $K(x_i) - c_i = 0$. Since c_i represents the number of quadrilateral corners of size i, $c_i \leq 4$. Therefore, we can conclude that $K \leq 4$ in order to satisfy $Kx_i - c_i = 0$. But contradicts this Proposition 8. Therefore,

$$Kx_i - c_i > 0 \Longrightarrow h_i \neq 0.$$

Hence, we see that there exist at least one i-hub for $i = l, m$, and n. ■

3.3 Some restrictions

There are only a finite number of (l, m, n)-triangles that can tile a quadrilateral so that the corner condition is satisfied. Since there must be at least one l-hub, m-hub and n hub each, then we must have l, $m, n \leq K$ and thus $l, m, n \leq 60$. We can get much better bounds however. Assume that $l \leq m \leq n$. Then, $s, t, u, v \leq n$ and

$$\mu(s, t, u, v) = 2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v} \leq 2 - \frac{1}{n} - \frac{1}{n} - \frac{1}{n} - \frac{1}{n} = \mu(n, n, n, n).$$

It follows that

$$l, m, n \leq K \leq \frac{\mu(n, n, n, n)}{\mu(l, m, n)} < \frac{2}{\mu(l, m, n)}. \tag{12}$$

Thus, for instance, for a $(2, 3, n)$ triangle we have

$$n \leq \frac{2 - \frac{4}{n}}{\frac{1}{6} - \frac{1}{n}} = 12\frac{n-2}{n-6}.$$

Solving this inequality for integer solutions, we obtain $3 \leq n \leq 16$. Since the triangle is hyperbolic then we further restrict $7 \leq n \leq 16$. Analogously, for $(2,4,n)$ triangles we get $5 \leq n \leq 10$, and for $(3,3,n)$ triangles we have $4 \leq n \leq 7$.

The inequality 12 combined with $K \geq 6$ (Proposition 8) shows that we get an area restriction $\mu < \frac{1}{3}$. We now prove a stronger restriction $\mu \leq \frac{1}{4}$, which allows us to select all our μ-data from the table in Appendix A.

Proposition 10 *If an (l,m,n)-triangle subdivides a quadrilateral, then $\mu(l,m,n) \leq \frac{1}{4}$.*

Suppose that $\mu = \mu(l,m,n) > \frac{1}{4}$. Let $\nu = \mu(s,t,u,v)$, we know that $\nu \leq 2$ and so satisfies.

$$K = \frac{\nu}{\mu} \Longrightarrow K < 8.$$

Since K must be an integer, $K \leq 7$. From Proposition 9, we know that there exists at least one l-hub, one m-hub, and one n-hub. Thus

$$K \leq 7 \Longrightarrow l,m,n \leq 7.$$

WLOG, we may assume $l \leq m \leq n \leq 7$. We consider four cases.

Case 1 $n = 7$, $K = 7$ There exists exactly one 7-hub. Since no other triangles can be added the remaining angles must both equal $\frac{\pi}{2}$ in order to be a quadrilateral. But then the resulting $(2,2,7)$-triangle is not hyperbolic.

Case 2 $n = 6$, $K = 6,7$ As above, there is exactly one 6-hub. At most one other triangle can be added and thus either 2 or possibly 3 triangles meet at the vertices. Since there is at least one hub of each type, then the possible triangles are $(2,2,6)$, $(2,3,6)$, and $(3,3,6)$. The first two are not hyperbolic and the last one does not satisfy $\mu < \frac{1}{3}$.

Case 3 $n = 5$, $K = 6,7$ Again there is exactly one 5-hub and at most two triangles can be added. Thus, except at the 5-hub, 2,3 or 4 triangles meet at each vertex. The only possibilities are $(2,2,5)$, $(2,3,5)$, $(2,4,5)$, $(3,3,5)$, $(3,4,5)$, and $(4,4,5)$ triangles. The first two are not hyperbolic, and for the last two the upper bound $\frac{\mu(n,n,n,n)}{\mu(l,m,n)}$ for K is smaller than 6. The middle two triangles satisfy the bound.

Case 4 $n = 4$, $K = 6, 7$ The only possible triangles are $(3,3,4)$, $(3,4,4)$, and $(4,4,4)$ triangles, all of which satisfy the inequality.

Since there are no hyperbolic triangles with $n \leq 3$, the proof is complete.

■

3.4 Computer algorithm for finding quadrilaterals using Maple

Having found an (l, m, n) and an (s, t, u, v) such that all above restrictions hold, a quadrilateral tiled by the triangle is sought out using a computer program developed in Maple. This program is given an (l, m, n)-triangle, and a quadruple (s, t, u, v) and tests to see if a corresponding quadrilateral can be drawn along the lines of the triangle tiling. Since the triangle tiling exists, a quadrilateral tiling can be shown to exist by drawing a boundary with four sides along lines of the tiling and satisfying the corner condition.

We now describe the algorithm; look at Figures 5, 6 and 7 in the next section or in the figures in the section 8 to help understand how the algorithm works. The algorithm starts by picking an angle with measure $\frac{\pi}{s}$ in the master tile. Pick an edge of the angle and move out along the ray determined by this edge. We move along the ray and observe the triangles we meet on the same side of the ray as our original triangle. We stop at a triangle which meets the ray in an edge whose *second vertex* encountered is at a $\frac{\pi}{t}$ angle of the triangle. To get the second side we turn the corner through $\pi - \frac{\pi}{t}$ radians toward the side of the original triangle and proceed along the next ray in the tiling and stop at a vertex of size $\frac{\pi}{u}$. Turn the corner through $\pi - \frac{\pi}{u}$ in the same direction as the first turn and produce a third side and so on. When we have finished, if the end of the fourth segment is the beginning of the first segment then we have a quadrilateral, otherwise we don't. Note that there are only a finite number of possibilities for each side of the quadrilateral since the number of hubs on the edges is bounded by the hub numbers.

For each of the possible quadrilateral boundaries the program creates four edge words in p, q, and r by making reflections through the various hubs along an edge of the quadrilateral. The four edge words are concatenated to form a *boundary word.* The four edges close up to form a quadrilateral if and only if the boundary word reduces to the identity. The pattern of hubs along the boundary allow us to quickly write down the boundary word. The details of determining the boundary word and showing it is the identity is described in detail in some sample calculations in the next section.

3.5 Proving the boundary word is the identity.

By attempting to draw a quadrilateral around the triangle tiling, as above, we have a word which we want to prove is the identity. There are two possible approaches. The first approach is see if the word can be reduced to the identity, using the relations that we have in the group T^*. Though in any specific example it always seems possible to find a reduction to the identity (when it exists) the authors decided against attempting to develop a computer algorithm to decide whether a word was reducible to the identity partly because of the apparent difficulty of doing so suggested by [3] p. 672, and because of the easy implementation of the second approach. The approach used in this case converts the product of an even number of reflections into a matrix by means of a homomorphism

$$q : T \to PSL_2(\mathbb{C}),$$

which we describe shortly. The matrix images are computed numerically using Maple, which introduces some error into our calculations due to rounding off the entries in the matrices. This can be remedied that noting that if the computations are carried out with great accuracy, say to 50 decimal places as done in the study, then a product which is not the identity will not be close to the identity. For each boundary word found to be close to the identity the picture of the tiling of the quadrilateral was directly constructed using a Matlab program.

We finish this section by describing the map q and stating a theorem that guarantees that our conclusions based on approximate calculations are justified. The reflections in the sides of an (l, m, n) triangle are inversions in the circle defining the side (or reflection in a diameter). Every such reflection has the form $T_A \circ \varepsilon$ where

$$T_A(z) = \frac{az+b}{cz+d}, \ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$\epsilon(z) = \overline{z}.$$

The matrix is determined only up to a scalar multiple. If the inversion is in a circle centered at z_0 and perpendicular to the unit disc then A has the form

$$A = \begin{bmatrix} z_0 & -1 \\ 1 & -\overline{z_0} \end{bmatrix}.$$

Otherwise, the line is a diameter meeting the positive x-axis at the angle θ and A has the form.

$$A = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

If two of the edges of the (l, m, n) triangle are diameters through the origin, then, the third side is easily determined using analytic geometry. Let P, Q, R be the matrices such that the reflections p, q, r are associated to:

$$\begin{aligned} p &\leftrightarrow T_P \circ \varepsilon, \\ q &\leftrightarrow T_Q \circ \varepsilon, \\ r &\leftrightarrow T_R \circ \varepsilon. \end{aligned}$$

The elements $a = pq$, $b = qr$, and $c = rp$ are associated to:

$$\begin{aligned} a &\leftrightarrow T_{P\overline{Q}}, \\ b &\leftrightarrow T_{Q\overline{R}}, \\ c &\leftrightarrow T_{R\overline{P}}, \end{aligned}$$

since

$$\begin{aligned} \varepsilon \circ T_A \circ \varepsilon &= T_{\overline{A}}, \text{ and} \\ T_{AB} &= T_A \circ T_B, \end{aligned}$$

where $\overline{A}$ is obtained by conjugating the entries of A.

Now a word w in p, q, r in T^* representing the identity must have an even number of factors since an odd word is orientation reversing. Thus w is a word in $a = pq$, $b = qr$, $c = rp$, $a^{-1} = qp$, $b^{-1} = rq$, and $c^{-1} = pr$. Thus the matrix corresponding to w is obtained by making substitutions based on:

$$\begin{aligned} a &\to P\overline{Q}, \\ b &\to Q\overline{R}, \\ c &\to R\overline{P}. \end{aligned} \tag{13}$$

The following theorem now guarantees when an element is approximately equal to the identity is actually equal to the identity. We first define the L^∞ norm on matrices:

$$\|A\| = \max_{k,l}(|A(k, l)|)$$

Proposition 11 *Let T be the group of orientation-preserving isometries generated by an (l, m, n) triangle Δ in the hyperbolic plane. Let $A \in PSL_2(\mathbb{C})$ be a matrix representing an element of $g \in T$, according to the substitution in 13. Then, there is a ϵ_Δ, depending on Δ and not on A, such that if $\min(\|A - I\|, \|A + I\|) \le \epsilon_\Delta$ then g is the identity in T.*

The proposition gets used in the following way. Suppose that the matrix A corresponds to g as in proposition and that A is computed approximately as $\overline{A}$. Suppose further that $\left\|\overline{A} - I\right\| \leq \frac{\epsilon_\Delta}{2}$ by calculation and that $\left\|A - \overline{A}\right\| \leq \frac{\epsilon_\Delta}{2}$ by controlling the accuracy of the computation. Then, $\|A - I\| \leq \left\|A - \overline{A}\right\| + \left\|\overline{A} - I\right\| \leq \epsilon_\Delta$. and so that g is the identity.

Remark 12 *The proof of the theorem and the discussion of the method for estimating* $\left\|A - \overline{A}\right\|$ *are beyond the scope of this technical report and will appear in a subsequent publication. A "back of the envelope" calculation showed that 50 digits is adequate.*

4 The search for divisible tilings

Here is our main result on divisible tilings satisfying a corner condition.

Theorem 13 *Suppose the (l, m, n) triangle Δ tiles the (s, t, u, v) quadrilateral Ω so that the corner condition is satisfied. Then Δ and Ω must form one of the thirteen cases in Table 1 of section 8. A picture of the tiled quadrilateral is given in Table 2 section 8.*

The rest of this section gives an overview of the search procedure and some sample calculations for finding quadrilaterals or showing none exist. The overall search has these steps:

1. Enumerate all of the (l, m, n) triangles for which $l \leq m \leq n$, $\mu(l, m, n) \leq \frac{1}{4}$ and $l, m, n \leq 60$, and $n \leq (2-4n)/\mu(l, m, n)$. The table of branching data in Appendix A can be used here.

2. For each triangle in 1 enumerate all candidate quadruples (s, t, u, v) formed from l, m, n.

3. For each candidate pair of an (l, m, n) and an (s, t, u, v) select those which pass the K-test and the hub tests, i.e., the quotients calculated in 9 and 11 must be integers.

4. For each candidate pair resulting from 3, perform the quadrilateral search algorithm.

Remark 14 *The steps 2, 3 and 4 can be easily modified to include tilings not satisfying the corner condition, though the l, m, n are no longer bounded.*

4.1 Failed subdivisions

Let us first look at some failed subdivisions.. We found many examples which did not produce an integer K or h_i.

Let us try to subdivide a $(7,7,7,7)$-quadrilateral with a $(2,4,7)$-triangle. We find that

$$\mu = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{7} = \frac{3}{28}.$$

We then find ν the area of the quadrilateral, divided by π:

$$\nu = 2 - \frac{1}{7} - \frac{1}{7} - \frac{1}{7} - \frac{1}{7} = \frac{10}{7}.$$

Now we can calculate K and find that

$$K = \frac{\nu}{\mu} = \frac{\frac{10}{7}}{\frac{3}{28}} = \frac{40}{3}.$$

Clearly, K is not an integer. Therefore, we can conclude that the $(2,4,7)$-triangle does not subdivide a $(7,7,7,7)$-quadrilateral because it failed the K-test.

Now let us examine the case of attempting to tile a $(9,9,9,9)$-quadrilateral by a $(2,3,9)$-triangle. Calculations reveal that $K = 28$. Therefore, we now proceed to the h_i-tests. We find that

$$h_n = \frac{K(x_n) - c_n}{n} = \frac{28(1) - 4}{7} = \frac{24}{7}.$$

We see that h_n is not an integer. Therefore, we can conclude from the failed h_n-test that the $(2,3,9)$-triangle does not subdivide a $(9,9,9,9)$-quadrilateral.

The above two examples clearly failed at least one of two requirements for a subdivision to exist: the K-test *and* the hub tests. However, there exist quadrilaterals which cannot be subdivided by triangles for which K and all h_i are integers. Consider the $(2,3,7)$-triangle and the $(7,7,7,7)$-quadrilateral. After calculations, we see that $\mu = \frac{1}{42}$, $\nu = \frac{10}{7}$, and $K = 60$. It was also found that $h_2 = 30$, $h_3 = 20$, and $h_7 = 8$. Thus far, it appears that the $(2,3,7)$ will subdivide the $(7,7,7,7)$ quadrilateral. However, our algorithm reveals that it is impossible for a $(2,3,7)$ to subdivide the $(7,7,7,7)$.

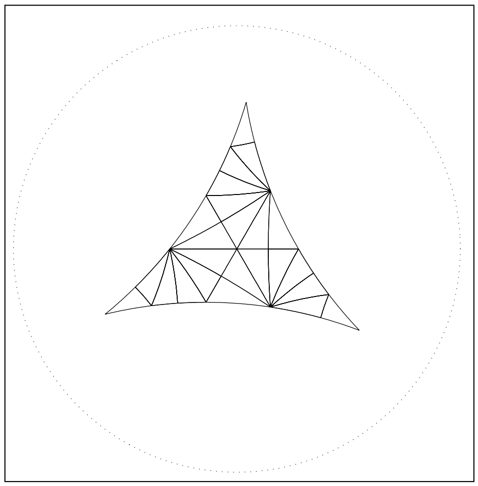


Figure 5. $(2,3,7)$ tiling of $(7,7,7)$

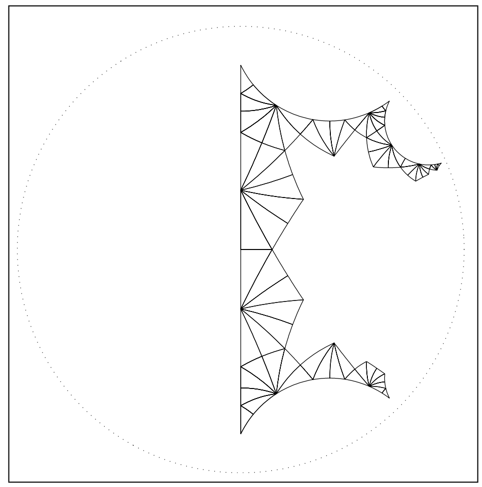


Figure 6. A failed boundary search.

It is helpful to examine this case in a bit more detail. Each of the four edges of any quadrilateral must have two 7-hubs. For, an edge with only one 7-hub and terminating $\frac{\pi}{7}$ angles must close up to a $(7,7,7)$ triangle as in Figure 5 above. Similarly, there cannot be zero 7-hubs, as Figure 5 also shows. Thus every edge must have exactly two 7-hubs, and hence there is a unique way to construct the quadrilateral. However, as Figure 6 shows, the attempt to construct such a quadrilateral clearly fails. Figure 6 also clearly illustrates the sequence of triangles used in constructing the boundary word. In the algorithm the boundary word is produced, and not the picture.

4.2 A successful subdivision

Now let us examine a quadrilateral which can be subdivided by triangles. Let us subdivide the $(5,5,5,5)$-quadrilateral with the $(2,4,5)$-triangle. It is found that $\mu = \frac{1}{20}$, $\nu = \frac{6}{5}$, and $K = 24$. See Table 2 section 8 - for a picture of the $(5,5,5,5)$ subdivided by the $(2,4,5)$. The hub tests give $h_2 = 12, h_4 = 6$ and $h_5 = 4$. By looking at Figure 7, it is clear the any edge with out a 5-hub closes up to a triangle. Since $h_5 = 4$ then there must be exactly one 5-hub on each side. This greatly restricts the boundary searches. Let us consider the lower boundary tiles on the lower edge of the quadrilateral as pictured in Figure 7.

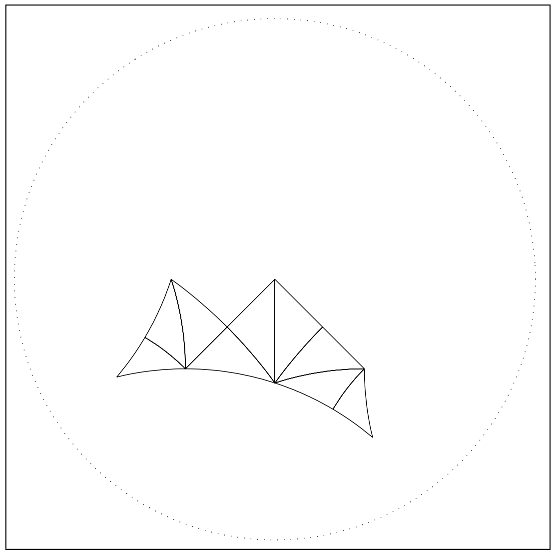


Figure 7. Boundary tiles of lower edge.

The far left triangle as oriented is a $(5, 4, 2)$-triangle. Though the order of l, m and n is unimportant for first three steps of the search, the ordering is very important in the quadrilateral construction phase. Thus for this particular triangle we have:

$$(pq)^5 = a^5 = 1,$$
$$(qr)^4 = b^4 = 1,$$
$$(rp)^2 = c^2 = 1.$$

Now let us construct the edge word corresponding to this part of the boundary. We label the triangles $\Delta_0, \Delta_1, \ldots, \Delta_8$ as we move from left to right. It is obvious that $\Delta_1 = r\Delta_0$ since we reflect over the r edge. The reflections in sides of Δ_1 are rpr, rqr and $r = rrr$. In fact if $\Delta' = g\Delta_0$ for some $g \in T^*$ the reflections in the sides of Δ' are gpg^{-1}, gqg^{-1} and grg^{-1}. From the picture we see that Δ_2 is the rqr image of Δ_1 and hence $\Delta_2 = (rqr)r\Delta_0 = rq\Delta_0$. Continuing one more step we see that the reflections in the sides of Δ_2 are $rqpqr$, $rqr = rqqqr$ and $rqrqr$. Now Δ_3 is obtained by reflecting Δ_2 over the common $rqrqr$ edge. Thus $\Delta_3 = rqrqr\Delta_2 = (rqrqr)rq\Delta_0 = rqr\Delta_0$. The pattern is now evident which we now state as a proposition. We omit the easy induction proof.

Proposition 15 *Let $\Delta_0, \Delta_1, \ldots, \Delta_s$ be a sequence of tiles such that Δ_i and Δ_{i-1} meet in edge e_i. Let r_i the unique edge among $\{p, q, r\}$ such that r_i matches e_i by the unique transformation in T^* carrying Δ_0 to Δ_i. Let r_i also denote the reflection in r_i. Then*

$$\Delta_i = r_1 \cdot r_2 \cdot \cdots \cdot r_i \Delta_0, \text{ for } i = 1, \ldots, s.$$

Applying the proposition we get

$$\Delta_8 = rqpqprqr\Delta_0 = w_1\Delta_0$$

Continuing on with the next edges we get:

$$\begin{aligned}
\Delta_{16} &= w_2 w_1 \Delta_0, \\
\Delta_{24} &= w_3 w_2 w_1 \Delta_0, \\
\Delta_{32} &= w_4 w_3 w_2 w_1 \Delta_0.
\end{aligned}$$

where $w_1 = w_2 = w_3 = w_4 = rqpqprqr$. The fact that all the edge words w_1, w_2, w_3, w_4 are equal is a consequence of symmetry of the quadrilateral. Observe that the edgewords are completely determined by the initial orientation of Δ_0 and the number and type of hubs we pass through along an edge. The process of finding the word can be sped up by observing that passing through a hub along the edge adds a well-defined subword to the edge word. thus one multiplies together a sequence of "hub words".

The boundary word we have is $(rqpqprqr)^4$. The candidate boundary closes up if and only if and only if $\Delta_{32} = \Delta_0$, i.e., if and only if $(rqpqprqr)^4\Delta_0 = \Delta_0$. Now it is well-known that T^* acts simply transitively on the triangles of the tiling and hence the boundary closes up if and only if

$$(rqpqprqr)^4 = 1.$$

But we have

$$\begin{aligned}
(rqpqprqr)^4 &= rqpqprq(rr)qpqprq(rr)qpqprq(rr)qpqprqr \\
&= rqpqpr(qq)pqpr(qq)pqpr(qq)pqprqr \\
&= rqpq(prp)q(prp)q(prp)qprqr
\end{aligned}$$

Now $prpr = 1$ so $prp = r$ and likewise $qrqrqrq = r$. Thus we further obtain

$$\begin{aligned}
(rqpqprqr)^4 &= rqp(qrqrqrq)prqr \\
&= rq(prpr)qr \\
&= rqqr \\
&= 1.
\end{aligned}$$

We have exactly computed the expected reduction. Alternatively, we could have translated the boundary word into a word in a, b, c and compute the matrix image in $PSL_2(\mathbb{C})$. One simply takes the letters in boundary word two at a time with the replacements: $p^2 = q^2 = r^2 = 1$, $pq = a$, $qr = b$, $pr = c$, $qp = a^{-1}$, $rq = b^{-1}$, and $pr = c^{-1}$. Our word and the replacement matrices are (to 5 decimal places):

$$(rqpqprqr)^4 = (b^{-1}ac^{-1}b)^4.$$

$$A \doteq \begin{bmatrix} .80903 + .58778i & 0 \\ 0 & .80903 - .58778i \end{bmatrix}$$
$$B \doteq \begin{bmatrix} .70708 + .97325i & -.66876i \\ .66876i & .70708 - .97325i \end{bmatrix}$$
$$C \doteq \begin{bmatrix} -1.2030i & -.39308 + .54105i \\ -.39308 - .54105i & 1.2030i \end{bmatrix}$$

The test matrix is seen to be approximately a scalar matrix, mapping to the identity in $PSL_2(\mathbb{C})$:

$$(B^{-1}AC^{-1}B)^4 \doteq \begin{bmatrix} -.9999 & 0 \\ 0 & -.9999 \end{bmatrix}.$$

The matrix calculations are done to 5 decimal places to ensure clarity. By computing to a greater number of digits, enough accuracy can be achieved to conclude that the boundary word is the identity. The matrix method is preferable since the computer (i.e., programmer) need not get creative about how to reduce words.

5 Symmetry patterns of the tiled quadrilaterals

A comparison between those quadrilaterals which could be subdivided by triangles and those that could not be subdivided led to an examination of the symmetry. It was found that the candidate quadruples (s, t, u, v) for which it was possible to subdivide by (l, m, n) triangles where l, m, n are distinct were all of the form (s, s, s, s), (s, t, s, t), or (s, s, t, t). Each of the quadrilaterals of this form support symmetries of reflection or rotation, while it would be impossible for quadrilaterals of the form (s, s, s, t) or (s, s, t, u) to support these symmetries. (Note that because we are concerned only with quadrilaterals with single triangles at the vertices, each s, t, u, v is equal to an l, m, or n). Therefore, we formed the following conjecture. which is now a corollary of the classification result.

Corollary 16 *For quadrilaterals subdivided by scalene (l, m, n)-triangles (distinct l, m, n), such that the corner condition is satisfied, (s, t, u, v) cannot be of the form (s, s, s, t) or (s, s, t, u), or any permutation of these where s, t, u are all distinct and $s, t, u \in \{l, m, n\}$.*

If the triangle is isosceles the above corollary needs to be interpreted differently, as the following example shows. Every tile is congruent to the master tile by a unique element of T^*. Thus every vertex has a unique type depending on which vertex of the master tile it is congruent to. Let us examine the $(4, 3, 3)$ tiling of a $(4, 3, 4, 3)$ given in the catalogue (Case 5). Label the vertices of the triangle R, P, Q as we move counterclockwise along the boundary starting at the $\frac{\pi}{4}$ vertex. The types of the quadrilateral vertices starting counterclockwise at R are then R, Q, R, P. It still has the form (s, t, s, t) but all three types of vertices occur. In the scalene case at most two vertex types occur.

All of the quadrilaterals in the catalogue exhibit a symmetry of order 2 some of order 4. Note that the corollary above follows from this observation. Note also that in the isosceles cases the internal symmetry of the triangle leads to a symmetry of the triangle which is not in T^* and that it is only the isosceles cases that have a symmetry not in T^*.

Theorem 17 *If a quadrilateral Ω can be subdivided by an (l, m, n)-triangle such that the corner condition holds, then Ω supports either 180° rotation or a reflection. If the symmetry lies in T^*, then the number of triangles, K, which subdivide the quadrilateral is even.*

Proof. The theorem may be proven by the somewhat unsatisfying method of visually verifying the each quadrilateral has such a symmetry. Now the order of the symmetry group of the quadrilateral must be $2, 4$, or 8 since it divides the order of the symmetry group of the square and is not trivial. Thus all orbits are even unless some triangle is fixed by all elements of the symmetry group. But if one of the non-trivial symmetries is in T^* then we have a contradiction since no triangle is fixed by a non-trivial element of T^*. If every orbit is even then there are an even number of triangles. ■

Let us examine each case of a rotation or reflection in greater detail.

Case 1: Reflection If the reflection is in T^* then there is an edge e, along which Ω can be reflected. Let N represent the number of tiles which

subdivide S that are only on one side of e. Each tile, t, is reflected through e to a tile on the other side of N. Therefore, there exist $K = 2N$ tiles which subdivide the quadrilateral S. If the triangle is isosceles then it is possible that the reflection line is the symmetry line of a triangle fixed by the reflection. In this case the remaining triangles occur in mirror image pairs and the number of triangles is odd. This happens in only one case.
Case 2: Rotation There exists a point of rotation, x, through which the quadrilateral, Ω can be rotated to maintain symmetry. The rotation must have order 2 or 4, since it rotates a quadrilateral. It cannot fix an interior point of a triangle since it would then have order 3. Thus the number of triangles is divisible by 2 or 4. Both cases occur. If the rotation is in T^* then there can be at most two types of vertices at the corners of the quadrilateral, since T^* preserves the type of vertices. In the case of the $(4,3,3)$ tiling of $(4,3,4,3)$ discussed above the rotational symmetry is not in T^*.

6 Surfaces which support both tilings

Now that we have found the possible subdivided tilings, we would like to determine the kind of surfaces on which these subdivided tilings can exist. The surfaces of interest are those in which both the tiling by quadrilaterals and the tiling by triangles are geodesic, kaleidoscopic tilings on the surface. Our task is to find the such surfaces of smallest genus or area. Because of the Riemann Hurwitz equation the minimal genus by determining the tiling group of the minimal surfaces. A formal statement of the result is the following theorem.

Theorem 18 *Suppose the* (l,m,n) *triangle* Δ *tiles the* (s,t,u,v) *quadrilateral* Ω *so that the corner condition is satisfied. Then there is a unique surface of minimal genus* S *which supports a divisible tiling with tiling pair isometric to* (Δ,Ω). *The genus and a description of the tiling group are given in Table 1 of section 8.*

6.1 Universal cover

The subdivided tilings were found in $\mathbb{H}$ which is the *universal covering space* of our surfaces. By constructing the tilings in this plane first, we can then map the tilings onto a surface of genus $\sigma \geq 2$. Given a tiling of a surface S there is a tiling of $\mathbb{H}$ and a map $\pi : \mathbb{H} \to S$ mapping the tiling of the plane onto the tiling of the surface. Let us sketch how this map may be constructed by means of the tiling and the map $T^* \to G^*$. Recall that the generators of T^* and G^* satisfy similar relations. To distinguish them we let p, q, r denote the generators of T^* and $\overline{p}, \overline{q}, \overline{r}$ be the generators of G^*. The relations for p, q, r and $\overline{p}, \overline{q}, \overline{r}$ have the same form and T^* satisfies only those relations (see 5). Hence, there is a surjective map $\xi : T^* \to G^*$ given by

$$\xi(p) = \overline{p},\ \xi(q) = \overline{q},\ \xi(r) = \overline{r}.$$

Also let Δ_0 and $\overline{\Delta}_0$ denote the master tiles in $\mathbb{H}$ and in S. Now we define $\pi : \mathbb{H} \to S$ tile by tile as follows. If $\Delta = g\Delta_0$ is any tile in $\mathbb{H}$ let the image tile $\pi(\Delta)$ be defined by:

$$\overline{\Delta} = \pi(\Delta) = \overline{g}\overline{\Delta_0} = \xi(g)\overline{\Delta_0}.$$

There are a few details to verify but the map is well-defined and illustrates the geometric notion of covering the surface. Let Γ be the subgroup of T^* mapping to the identity in G^*. It is easy to show that Γ-equivalent tiles map to the same tile in S, and so S is the quotient $\mathbb{H}/\Gamma$. In fact, there is an isometry $h : \mathbb{H}/\Gamma \backsimeq S$. This isometry h maps the tiling on $\mathbb{H}/\Gamma$ to the tiling on S. We also note that Γ is a normal, fixed point free subgroup of orientation preserving elements of T^*, i.e., it has no rotations fixing a point of $\mathbb{H}$. Otherwise, the map $\xi : T^* \to G^*$ will not preserve the order of rotations which it clearly does. It can be shown that there is an (essentially) 1-1 correspondence between (l, m, n)-tiling groups and normal, fixed point free, orientation preserving subgroups of $T^* = T^*_{(l,m,n)}$.

6.2 Finding a Γ

Now let us bring the quadrilateral tiling into play. Let $\Omega_0 \supset \Delta_0$ be the master tile for the quadrilateral tiling on $\mathbb{H}$. The triangle Δ_0 generates the full triangle tiling group $T^* = T^*_{(l,m,n)}$, and $T = T_{(l,m,n)}$ is the subgroup of orientation preserving isometries of the triangle tiling group. Similarly, Ω_0 generates $Q^* = Q^*_{(s,t,u,v)}$, the full quadrilateral tiling group, and $Q = Q_{(s,t,u,v)}$ is the subgroup of orientation preserving isometries of the quadrilateral tiling group. Since our Δ and Ω tilings were both constructed in the universal cover, it is necessary to find a Γ such that both tilings are mapped to kaleidoscopic tilings on some surface S. For this to be true, it is necessary and sufficient for Γ to be a normal subgroup in T^*, T, Q^*, and Q. Since this is an important statement let us make a formal proposition of it. It is not to hard to prove though we don't do it.

Proposition 19 *Let $\Delta_0 \subset \Omega_0$ be the master tiles of kaleidoscopic, geodesic tilings of $\mathbb{H}$ of types (l, m, n) and (s, t, u, v) respectively. Let T^*, T, Q^*, and*

Q be the groups defined above. Then the correspondence $\Gamma \leftrightarrow \mathbb{H}/\Gamma = S$ is a correspondence between the fixed point free, orientation preserving subgroups normal in all of the groups T^*, T, Q^*, and Q and the kaleidoscopic, geodesic (s,t,u,v)-tilings on surfaces S which may be subdivided into a kaleidoscopic, geodesic (l,m,n)-tiling such that a master tile pair $\overline{\Delta_0} \subset \overline{\Omega_0}$ in S is isometric with the master tile pair $\Delta_0 \subset \Omega_0$ in $\mathbb{H}$. Furthermore, if $G \subset G^*$, and $H \subset H^*$ are the tiling groups on S generated by $\overline{\Delta_0}$ and $\overline{\Omega_0}$ respectively, then

$$G^* = T^*/\Gamma,\ G = T/\Gamma,\ \text{and}$$
$$H^* = Q^*/\Gamma,\ H = Q/\Gamma.$$

We would like Γ to be as large as possible, so that the tiling group we are looking at will be as small as possible. This minimal condition on the size of the tiling group G^* is desirable since we are looking for the smallest genus on which a subdivided tiling is possible. It is easy to see that by taking subgroups Γ of this largest group Γ_0 (assuming it exists) which are normal in T^*, T, Q^*, and Q, we can derive all other possible genera on which the divisible tiling is possible. Thus by finding the smallest genus surface that the divisible tiling occurs, we will have made it possible to find all possible genera.

Since we know that $\Gamma \subseteq Q$ and $\Gamma \lhd T^*$, it follows that $\Gamma \subseteq \bigcap_{x\in T^*}(xQx^{-1})$. So let us simply define

$$\Gamma_0 = \bigcap_{x\in T^*}(xQx^{-1}).$$

If we knew Γ_0 was torsion free then we would be done. In fact we have.

Proposition 20 *Let $\Delta_0 \subset \Omega_0$ be the master tiles of kaleidoscopic, geodesic tilings of $\mathbb{H}$ of types (l,m,n) and (s,t,u,v) respectively. Assume that the tilings satisfy the corner condition. Let T^*, T, Q^*, and Q be the tiling groups defined above. Then the subgroup*

$$\Gamma_0 = \bigcap_{x\in T^*}(xQx^{-1}).$$

is a fixed point free group acting on $\mathbb{H}$.

Though we don't have a general proof for the fixed-point free property of Γ_0, we can show Γ_0 is fixed point free for each example by calculation and the following proposition.

Proposition 21 *Suppose that $\Gamma \trianglelefteq T$, and that the images $\overline{a}$, $\overline{b}$ and $\overline{c}$ in T/Γ of generators a, b, and c of T have the same orders respectively. Then Γ is fixed point free.*

Proof. The group Γ has a fixed point if and only if some rotation $g \in \Gamma$ fixes a point $x \in \mathbb{H}$. But then as $g \in T$ it must fix some vertex of the tiling. It follows then that some T-conjugate of g fixes a vertex of the master tile. Since Γ is normal, we may assume without loss of generality that g itself fixes a vertex of the master tile, say the vertex fixed by a. But then $g \in \langle a \rangle$ and the order of $\overline{a}$ is given by:

$$o(\overline{a}) = \frac{l}{|\langle a \rangle \cap \Gamma|} \leq \frac{l}{|\langle g \rangle|} < l.$$

Similar conclusions arise if g fixes one of the other vertices. ∎

Remark 22 *Note that the above proof shows that if Γ is not fixed point free then the tiling on $\mathbb{H}/\Gamma$ will not be an (l, m, n) tiling, but rather an $(o(\overline{a}), o(\overline{b}), o(\overline{c})$-tiling..*

Remark 23 *We also have:*

$$\Gamma_0 = \bigcap_{x \in T} (xQx^{-1}).$$

To prove this note that $T^* = T \cup Ts$ where $s \in Q^* - Q$, is some reflection in the boundary of the master quadrilateral. But then, since $sQs^{-1} = Q$ then:

$$\begin{aligned}
\bigcap_{x \in T^*} (xQx^{-1}) &= \bigcap_{x \in T} (xQx^{-1}) \cap \bigcap_{x \in Ts} (xQx^{-1}) \\
&= \bigcap_{x \in T} (xQx^{-1}) \cap \bigcap_{x \in T} (xsQs^{-1}x^{-1}) \\
&= \bigcap_{x \in T} (xQx^{-1}).
\end{aligned}$$

6.3 Finding the genus

Now we are going to define Γ_0 in terms of the permutation representation of T on T/Q. Let π be the associated permutation representation of the action of T on the left cosets of Q in T. Then:

$$ker(\pi) = \bigcap_{x \in T} (xQx^{-1}) = \Gamma_0$$

since it is the intersection of all the stabilizer groups. The virtue of this approach is our ability to calculate the permutation representation of T using Magma. Very briefly, using Magma we can calculate the representing permutations $\pi(a), \pi(b)$, and $\pi(c)$ of a, b, and c on the coset space T/Q. If we also have

$$\begin{aligned} o(\pi(a)) &= o(a) = l, \\ o(\pi(b)) &= o(b) = m, \\ o(\pi(c)) &= o(c) = n, \end{aligned}$$

then Γ_0 is fixed point free. Thus the tiling on $S = \mathbb{H}/\Gamma_0$ has the same type and we get a divisible tiling on S. Since the tiling group $G \backsimeq \langle \pi(a), \pi(b), \pi(c) \rangle$, we can calculate the group order using Magma. Using the order of this group, and the information we know about the triangle tiling, we can find the genus of the tiled surface, using the Riemann-Hurwitz equation 4 for (l, m, n)-triangles. This equation can be rewritten as:

$$\sigma = 1 + \frac{|G|}{2} \left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right) \right). \tag{14}$$

Sample calculation Consider the tiling of a $(4, 4, 4, 4)$-quadrilateral by $(2, 4, 6)$-triangles. Using Magma, we find that the order of G, the group on the surface generated by the orientation-preserving isometries of the triangle, is 48. Using the Riemann-Hurwitz equation, we can see that:

$$\sigma = 1 + \frac{48}{2} \left(1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \right) \right) = 3.$$

Thus, the smallest surface on which a $(2, 4, 6)$-triangle can subdivide a $(4, 4, 4, 4)$-quadrilateral is on a surface of genus 3.

Let us show how we may set up this calculation for Magma. Magma can compute the "coset action" of one finitely presented group containing another. To this end let us recall that the T has the presentation.

$$T = \left\langle a, b, c : a^l = b^m = c^n = abc = 1 \right\rangle.$$

The subgroup Q has a similar presentation:

$$Q = \left\langle d, e, f, g : d^s = e^t = f^u = g^v = defg = 1 \right\rangle.$$

The generators d, e, f, g are counterclockwise rotations at the corners of the quadrilateral since they are products of reflections in successive edges. We know that the index of Q in T is finite, in fact it is K. Now it is just a simple matter to identify Q as a subgroup of T by finding each of the generators of Q as a word in a, b, c. Rather than describe the general process let us go through the steps in calculating the above example. From the catalogue we see that we need to orient our master triangle as a $(4, 2, 6)$ triangle in the lower left of a $(4, 4, 4, 4)$ quadrilateral. Now we get for free that $d = a$, since both master tiles share a common angle. The element $rpqpr$ carries the master tile to the next corner tile. Note that this edgeword maps the first corner vertex to the second corner vertex. Unfortunately, $rpqpr$ is not in T. However, $w_1 = ca^{-1}c = rpqprp$ is in T and still maps the first corner vertex to the second. It follows that $e = w_1 a w_1^{-1}$. Continuing along the second and third edges we find $w_2 = c^{-3}$, and $w_3 = ca^{-1}c$ such that $w_1 w_2$ and $w_1 w_2 w_3$ map the first vertex to the third vertex to the fourth vertex, respectively. Again the symmetry of the quadrilateral yields some symmetries in the words. Summarizing we have:

$$w_1 = ca^{-1}c,\ w_2 = c^{-3},\ w_3 = ca^{-1}c$$
$$d = a,\ e = w_1 a w_1^{-1},\ f = w_1 w_2 a w_2^{-1} w_1^{-1},\ g = w_1 w_2 w_3 a w_3^{-1} w_2^{-1} w_1^{-1}.$$

A Magma calculation gives:

$$\pi(a) = (2, 4, 8, 3)(5, 7, 6, 9),$$
$$\pi(b) = (1, 2)(3, 6)(4, 5)(7, 11)(8, 10)(9, 12),$$
$$\pi(c) = (1, 3, 7, 11, 5, 2)(4, 9, 12, 6, 8, 10).$$

and $|G| = 48$. It is obvious that orders are preserved and that the minimal tiling exists. It consists of 96 triangles or 8 quadrilaterals.

6.4 Structure of the tiling group

We now know the surfaces on which these divisible tilings exist. We would like to know more about the structure of the tiling group G on the surface. One way to find out more about the structure is to find the largest primitive group that is the image of some permutation representation of G. We chose this approach because the representation of T on T/Q is can be represented by the tiles within a singles quadrilateral. It turns out that the symmetries of the quadrilateral and its tiling have can be interpreted in terms of the primitivity of the G-action. A group G acting on a set A is called *primitive* if G is transitive and the only blocks in A are the trivial sets of size 1 and A itself. A *block* is defined as a non-empty subset B of A such that for all

$g \in G$ either $g(B) = B$ or $g(B) \cap B = \emptyset$. By partitioning A into blocks we obtain a permutation representation of G on the blocks. Geometrically, we partition the tiling of the quadrilateral into blocks via symmetries. By finding this large primitive image group (usually a symmetric or alternating group), we may better be able to study the structure of G. In fact if N is the kernel of the representation then we have

$$N \to G \to G/N,$$

and often N and G/N are well-known groups.

To find the largest primitive group, we look at the group G acting on a set $A = G/H$, and see if the image which is in $\sum_{G/H}$ is primitive. This amounts to seeing if there exists a subgroup lying in between the groups G and H. Let M be an intermediate subgroup lying between G and H. For the sake of argument we assume that $M = H \cup zH$ so that H has index 2 in M, i.e., M is minimal. The blocks are unions of cosets of the form $xH \cup xzH = xM$. Letting $y \in G$ act on H and M, we see that $yxM = yxH \cup yxzH$ so that G-action preserves the block structure. This shows that the image of the permutation representation, on G/H, call it G_1, is imprimitive. We can use Magma to find such minimal groups by looking for minimal partitions.

We are guaranteed an intermediate subgroup between G and H in nearly every case because of the symmetry we observed in the subdivided tilings. The symmetry corresponds to $|N_G(H)| > |H|$. If the tiling has symmetry of a 180 degree rotation, or a single reflection in an oval, then $|N_G(H) : H| = 2$. If the tiling has symmetry of a 90 degree rotation, or two reflections along ovals, then $|N_G(H) : H| = 4$. In either case, we are guaranteed an intermediate subgroup M. In only two cases is there no symmetry of the quadrilateral that yields an intermediate group. In those cases the symmetry of the quadrilateral does not come from G^*. In fact, by doing calculations in Magma, we can see that this is because, in this case, the group G_1 is primitive. A precise statement of the relationship between the normalizers and the symmetry group is given in the following proposition. We do not prove the proposition here since it is only used as a suggestion to look for primitive quotients of G.

Proposition 24 *Let Q, Q^*, T, T^* be the groups derived from a quadrilateral tiling subdivided by a triangular tiling. Let* $\mathrm{Stab}_{T^*}(\Omega_0)$ *denote the T^*-symmetry group of the quadrilateral. Then*

$$\mathrm{Stab}_{T^*}(\Omega_0) \backsimeq N_{T^*}(Q^*)/Q^* \backsimeq N_T(Q)/Q \backsimeq N_G(H)/H.$$

In the cases where we are guaranteed an intermediate subgroup exists, G_2, image of the permutation representation of G acting on the left cosets of H in G, may be partitioned into blocks. We now look at how G acts on these blocks ,and determine if the image of this action G_2, is primitive. In most cases, this group was primitive, but in three cases it was not. In the cases where $[N_G(H) : H] = 4$, G_2 will not be primitive since there is a subgroup chain $H \subsetneq M_1 \subsetneq M_2 \subsetneq G$.

For these cases where G_2 was not primitive, we again looked at the action on blocks. Since G_2 was not primitive, we know that there must exist some intermediate subgroup M_2 as above. Thus we can again partition G_2 into blocks and look at how these blocks are permuted by G_2. In all cases, by looking at the permutation representation, we can see that the image of the action is primitive. Thus, in only a few steps we can discover the largest primitive group in the image of G. These primitive groups were then classified using a list of primitive groups in Magma. A list of these groups, as well as the group $N_G(H)/H$ is available in Table 1, section 8. In all cases the number of intermediate groups was exactly as predicted by the structure of the group $N_G(H)/H$.

Also noted in the table are the kernels of the permutation representations that gave us the primitive groups. In many cases, the kernel is a somewhat large elementary abelian group. We hope to completely classify these groups G, giving good presentations of them, in further research.

7 Questions for further study

In very compact form we give a list of further topics of research on divisible tilings.

1. Relax the corner condition and find all divisible tilings.

2. Prove the symmetry results without using the classification.

3. Find necessary and sufficient geometric conditions to guarantee that Γ_0 fixed point free. Conjecture: Γ_0 is fixed point free if and only if all the hub numbers are positive.

4. Find and implement a reduction algorithm for words in the triangle groups.

5. Analyze the structure of the minimal genus tiling groups. The kernels of the permutation representations that gave primitive groups are

noted in the Tables in Section 8. In many cases, the kernel is a somewhat large elementary abelian group. Analyze these kernels and see if there are any other large groups when the corner condition is relaxed.

6. Describe the minimal quotients when Γ_0 is not fixed point free.

8 Catalogue of divisible tilings

This section contains a complete catalogue of all (s,t,u,v)-quadrilaterals which can be subdivided by (l,m,n)-triangles, and information on the minimum tiling. Some notes on the tables:

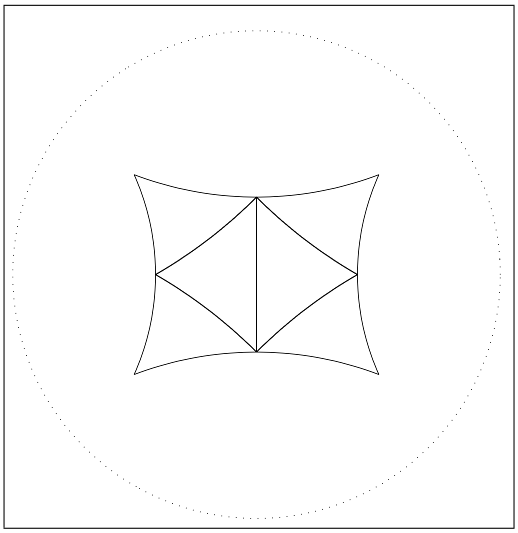
- The notation $(\mathbb{Z}_m)^n$ represents the direct-product of $\mathbb{Z}_m$ n-times.
- Although it is usual to arrange for $l \leq m \leq m$ the geometry of being included in the quadrilateral forces a different ordering. for both the triangle and the quadrilateral the ordering is obtained by moving about the figure in a counterclockwise sense, starting at the lower left corner of the quadrilateral.

Pictures of all of the quadrilaterals subdivided by triangles are also given in the second table in this section.

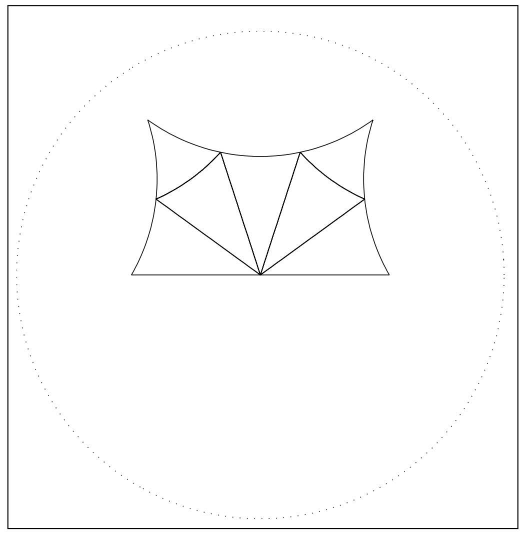
Table 1. Catalogue of Divisible Quadrilaterals

Case	(l,m,n)	(s,t,u,v)	K	Primitive Image	Kernel	$\|G\|$	σ	$N_G(H)/H$
1	$(4,4,3)$	$(4,4,4,4)$	6	S_3	$(\mathbb{Z}_2)^2$	$2^3 \cdot 3$	3	$\mathbb{Z}_2$
2	$(3,5,3)$	$(3,3,5,5)$	7	A_5	Identity	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	169	Identity
3	$(4,2,5)$	$(4,4,2,2)$	10	D_5	$(\mathbb{Z}_2)^4$	$2^5 \cdot 5$	5	$\mathbb{Z}_2$
4	$(4,2,5)$	$(4,2,4,2)$	10	S_5	$(\mathbb{Z}_2)^4$	$2^7 \cdot 3 \cdot 5$	49	$\mathbb{Z}_2$
5	$(4,3,3)$	$(4,3,4,3)$	10	A_{10}	Identity	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	75601	Identity
6	$(4,3,3)$	$(4,4,4,4)$	12	A_3	$(\mathbb{Z}_2)^2$,	$2^4 \cdot 3$	3	$\mathbb{Z}_4$
7	$(4,2,6)$	$(4,4,4,4)$	12	S_3	$(\mathbb{Z}_2)^3$	$2^4 \cdot 3$	3	$\mathbb{Z}_2 \times \mathbb{Z}_2$
8	$(5,2,5)$	$(5,5,5,5)$	12	$PSL(2,5)$	Identity	$2^2 \cdot 3 \cdot 5$	4	$\mathbb{Z}_2$
9	$(4,2,5)$	$(4,4,4,4)$	20	D_5	$(\mathbb{Z}_2)^4$	$2^5 \cdot 5$	5	$\mathbb{Z}_2$
10	$(4,2,5)$	$(4,4,5,5)$	22	S_{11}	$(\mathbb{Z}_2)^{10}$	$2^{18} \cdot 3^4 \cdot 5^2 \cdot 7$ $\cdot 11$	1021870081	$\mathbb{Z}_2$
11	$(5,4,2)$	$(5,5,5,5)$	24	$PGL(2,5)$	Identity	$2^3 \cdot 3 \cdot 5$	4	$\mathbb{Z}_4$
12	$(7,3,2)$	$(7,2,7,2)$	30	A_{15}	$(\mathbb{Z}_2)^{14}$	$2^{24} \cdot 3^6 \cdot 5^3 \cdot 7^2$ $\cdot 11 \cdot 13$	127529385984001	$\mathbb{Z}_2$
13	$(3,7,2)$	$(7,3,7,3)$	44	A_{21}	$(\mathbb{Z}_2)^{21}$	$2^{39} \cdot 3^9 \cdot 5^4 \cdot 7^3$ $\cdot 11^2 \cdot 13 \cdot 17 \cdot 19$	14030954608692 \ 056555520001	$\mathbb{Z}_2$

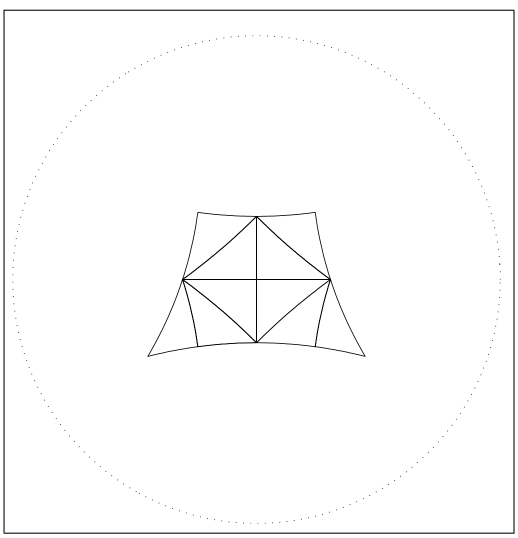
Table 2. Pictures of Divisible Quadrilaterals



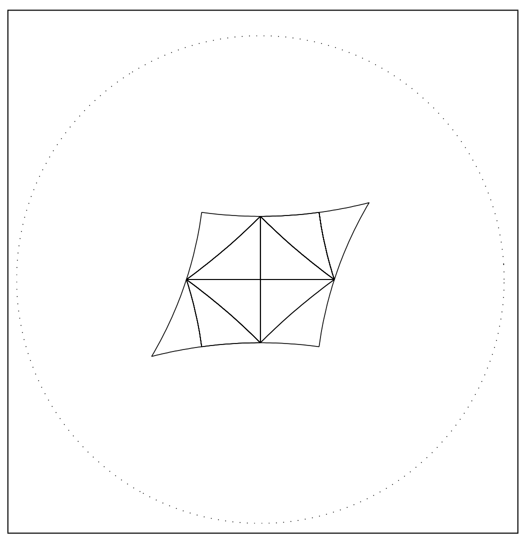
Case 1. 443 tiling of 4444 ($K = 6$)



Case 2. 353 tiling of 3355 ($K = 7$)

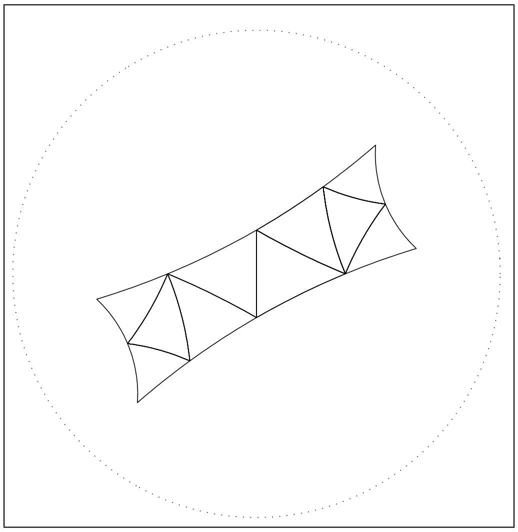


Case 3. 425 tiling of 4422 ($K = 10$)

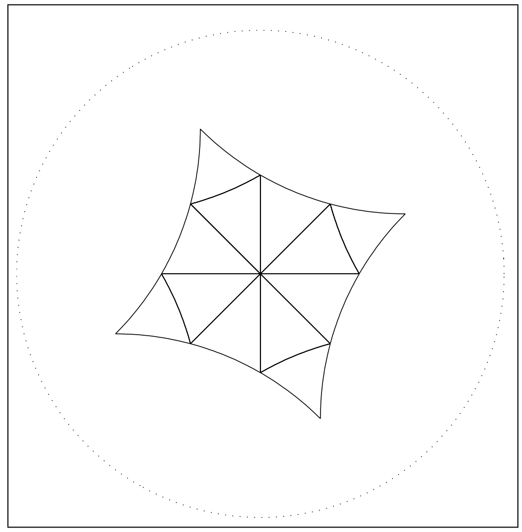


Case 4. 425 tiling of 4242 ($K = 10$)

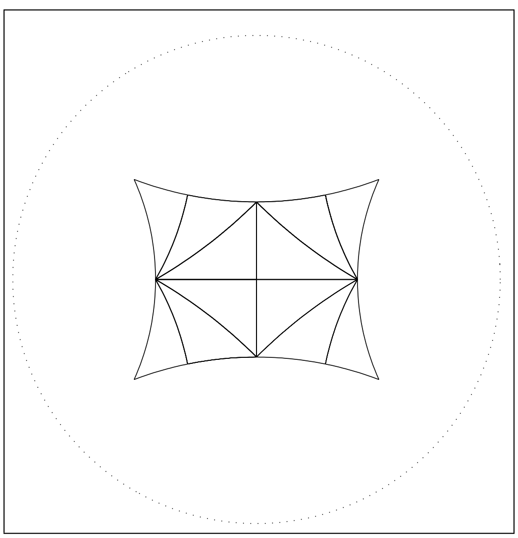
Table 2. continued



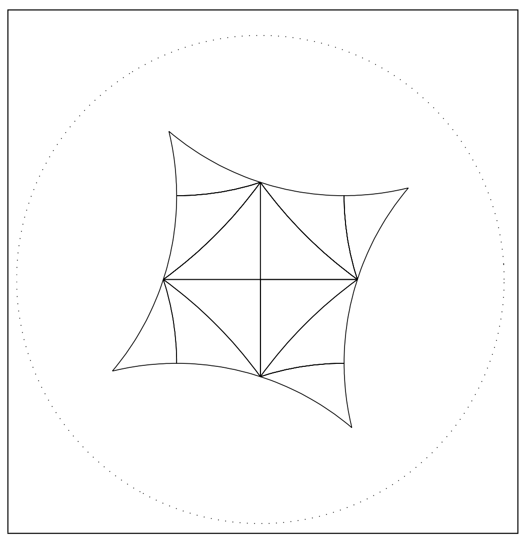
Case 5. 433 tiling of 4343 ($K = 10$)



Case 6. 433 tiling of 4444 ($K = 12$)

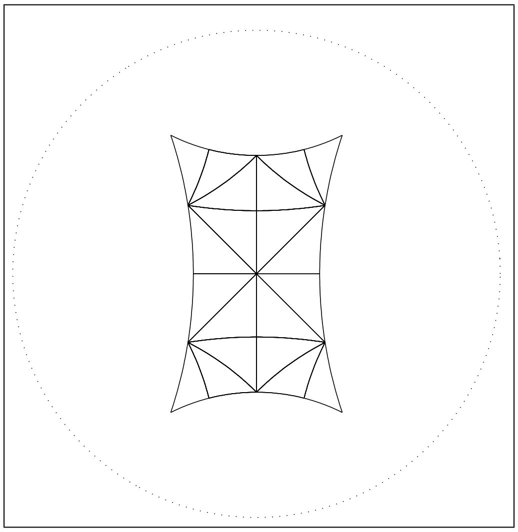


Case 7. 426 tiling of 4444 ($K = 12$)

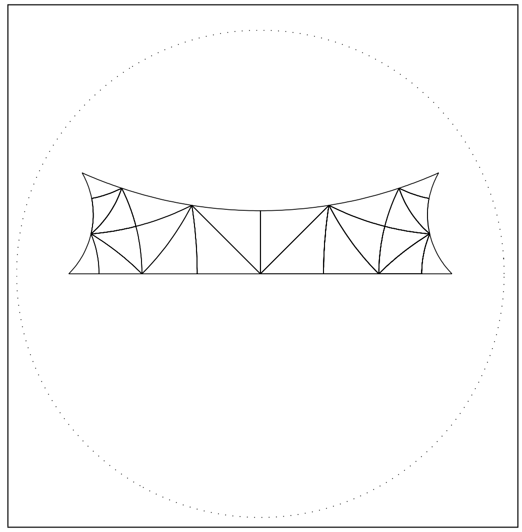


Case 8. 525 tiling of 5555 ($K = 12$)

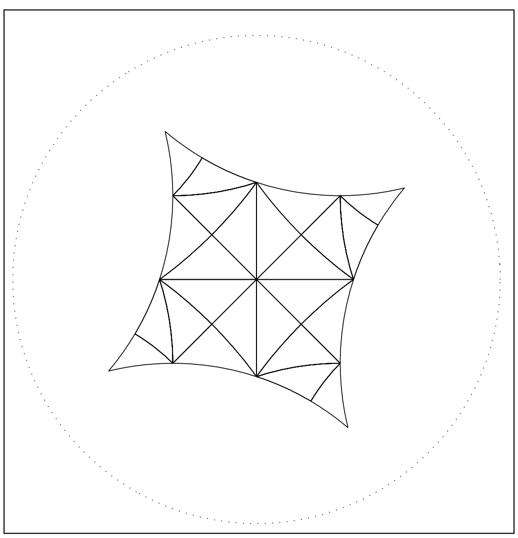
Table 2. continued



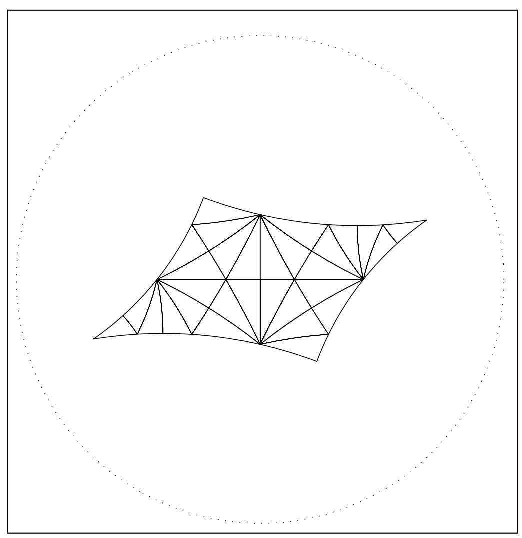
Case 9. 425 tiling of 4444 ($K = 20$)



Case 10. 425 tiling of 4455 ($K = 22$)

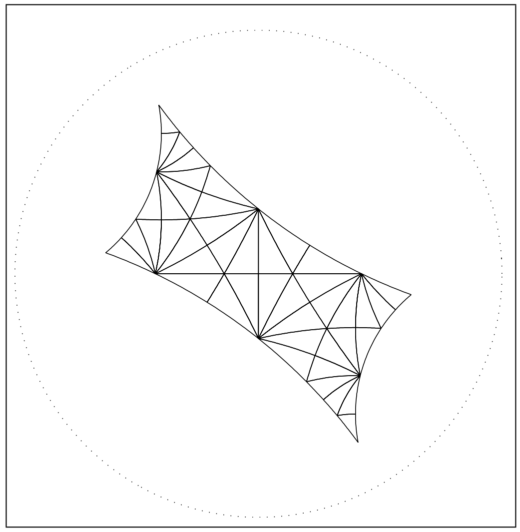


Case 11. 542 tiling of 5555 ($K = 24$)



Case 12. 732 tiling of 7272 ($K = 30$)

Table 2. continued



Case 13. 372 tiling of 3737 ($K = 44$)

Appendix A - Branching data table

Branching Data and $\mu(n_1, \ldots, n_t)$-values, $t = 3, 4, \mu \leq \frac{1}{4}$

$\mu(l, m, n) = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$ (triangles)

$\mu(k, l, m, n) = 2 - \frac{1}{k} - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$ (quadrilaterals)

$\mu(n_1, \ldots, n_t) = t - 2 - \sum_{i=1}^{t} \frac{1}{n_i}$ (t-sided polygon),

Branching Data	μ	**Branching Data**	μ	**Branching Data**	μ
$(2,3,7)$	$\frac{1}{42}$	$(2,3,d), d \geq 133$	$\frac{d-6}{6d}$	$(2,5,14)$	$\frac{8}{35}$
$(2,3,8)$	$\frac{1}{24}$	$(2,4,12), (2,6,6), (3,3,6)$	$\frac{1}{6}$	$(2,4,d), 47 \leq d \leq 55$	$\frac{d-4}{4d}$
$(2,4,5)$	$\frac{1}{20}$	$(3,4,4), (2,2,2,3)$	$\frac{1}{6}$	$(2,4,56), (2,7,8)$	$\frac{13}{56}$
$(2,3,9)$	$\frac{1}{18}$	$(2,4,13)$	$\frac{9}{52}$	$(2,4,d), 57 \leq d \leq 59$	$\frac{d-4}{4d}$
$(2,3,d), d = 10, a11$	$\frac{d-6}{6d}$	$(2,5,8)$	$\frac{7}{40}$	$(2,4,60), (2,5,15)$	$\frac{7}{30}$
$(2,3,12), (2,4,6), (3,3,4)$	$\frac{1}{12}$	$(2,4,d), 14 \leq d \leq 16$	$\frac{d-4}{4d}$	$(2,6,10), (3,3,10)$	$\frac{7}{30}$
$(2,3,d), d = 13, 14$	$\frac{d-6}{6d}$	$(2,5,9)$	$\frac{17}{90}$	$(2,4,d), 61 \leq d \leq 79$	$\frac{d-4}{4d}$
$(2,3,15), (2,5,5)$	$\frac{1}{10}$	$(2,6,7), (3,3,7)$	$\frac{4}{21}$	$(2,4,80), (2,5,16)$	$\frac{19}{80}$
$(2,3,16)$	$\frac{5}{48}$	$(2,4,d), 17 \leq d \leq 19$	$\frac{d-4}{4d}$	$(2,4,d), 81 \leq d \leq 113$	$\frac{d-4}{4d}$
$(2,4,7)$	$\frac{3}{28}$	$(2,4,20), (2,5,10)$	$\frac{1}{5}$	$(2,5,17)$	$\frac{41}{170}$
$(2,3,17)$	$\frac{11}{102}$	$(2,4,d), 21 \leq d \leq 23$	$\frac{d-4}{4d}$	$(2,4,d), 114 \leq d \leq 131$	$\frac{d-4}{4d}$
$(2,3,d), 18 \leq d \leq 23$	$\frac{d-6}{6d}$	$(2,4,24), (2,6,8), (3,3,8)$	$\frac{5}{24}$	$(2,4,132), (2,6,11), (3,3,11)$	$\frac{8}{33}$
$(2,3,24), (2,4,8)$	$\frac{1}{8}$	$(2,5,11)$	$\frac{23}{110}$	$(2,4,d), 133 \leq d \leq 179$	$\frac{d-4}{4d}$
$(2,3,d), 25 \leq d \leq 29$	$\frac{d-6}{6d}$	$(2,4,d), 25 \leq d \leq 27$	$\frac{d-4}{4d}$	$(2,4,180), (2,5,18)$	$\frac{11}{45}$
$(2,3,30), (2,5,6), (3,3,5)$	$\frac{2}{15}$	$(2,4,28), (2,7,7)$	$\frac{3}{14}$	$(2,4,d), 181 \leq d \leq 251$	$\frac{d-4}{4d}$
$(2,3,d), 31 \leq d \leq 35$	$\frac{d-6}{6d}$	$(2,4,29)$	$\frac{25}{116}$	$(2,4,252), (2,7,9)$	$\frac{31}{126}$
$(2,3,36), (2,4,9)$	$\frac{5}{36}$	$(2,4,30), (2,5,12), (3,4,5)$	$\frac{13}{60}$	$(2,4,d), 253 \leq d \leq 379$	$\frac{d-4}{4d}$
$(2,3,d), 37 \leq d \leq 59$	$\frac{d-6}{6d}$	$(2,4,d), 31 \leq d \leq 35$	$\frac{d-4}{4d}$	$(2,4,380), (2,5,19)$	$\frac{47}{190}$
$(2,3,60), (2,4,10)$	$\frac{3}{20}$	$(2,4,36), (2,6,9), (3,3,9)$	$\frac{2}{9}$	$(2,4,d), d \geq 381$	$\frac{d-4}{4d}$
$(2,3,d), 61 \leq d \leq 104$	$\frac{d-6}{6d}$	$(2,4,37)$	$\frac{33}{148}$	$(2,5,20), (2,6,12), (2,8,8)$	$\frac{1}{4}$
$(2,3,105), (2,5,7)$	$\frac{11}{70}$	$(2,5,13)$	$\frac{29}{130}$	$(3,3,12), (3,4,6), (4,4,4)$	$\frac{1}{4}$
$(2,3,d), 106 \leq d \leq 131$	$\frac{d-6}{6d}$	$(2,4,d), 38 \leq d \leq 46$	$\frac{d-4}{4d}$	$(2,2,2,4)$	$\frac{1}{4}$
$(2,3,132), (2,4,11)$	$\frac{7}{44}$				

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[4] MAGMA, John Cannon, University of Sydney, john@maths.usyd.edu.au

[5] MAPLE V, Waterloo Maple Inc., Waterloo, Canada.

[6] MATLAB, The Mathworks, Natick, Massachusetts.