Splitting tiled surfaces with abelian conformal tiling group

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Abstract

Let φ be a reflection on a closed Riemann S, i.e., an anti-conformal involutary isometry of S with a non-empty fixed point subset. Let S_φ denote the fixed point subset of φ, which is also called the *mirror* of φ. If $S - S_\varphi$ has two components then φ is called *separating* and that S splits at the mirror S_φ. Otherwise φ is called *non-separating*. We assume that the system of mirrors, S_ψ, as ψ varies over all reflections in the isometry group $\mathrm{Aut}^*(S)$ defines a tiling of the surface, consisting of triangles. In turn, the tiling determines a subgroup $G \subset \mathrm{Aut}^*(S)$ of conformal automorphisms of S. We give a simple criterion, derived from the geometry of the tiling, for determining whether the reflection is separating by means of equations in the rational group algebra of G. Examples for abelian G, where the computations are especially simple, are presented.

Contents

1 Introduction

Let S be a closed Riemann surface of genus $\sigma \geq 2$, with a hyperbolic metric of constant negative curvature -1. A symmetry φ of S is an anti-conformal, involutary isometry with respect to this metric. The fixed point set of φ, $S_\varphi = \{x \in S : \varphi(x) = x\}$, also called the *mirror* of φ, must be a disjoint union of circles each of which we call an *oval*. If the mirror is non-empty then φ is called a reflection in S_φ. In this case the complement $S - S_\varphi$ has either one or two components. We call φ *separating* if there are two components and *non-separating* otherwise. Alternatively we say that S *splits* at the mirror of φ. The set of all mirrors, S_ψ, as ψ varies over all reflections in the isometry group $\mathrm{Aut}^*(S)$, defines a tiling of the surface. A tiling generates a *tiling group,* G^*, generated by reflections. In turn there is a subgroup G, of index 2 in G^*, generated by rotations of S, which are conformal isometries. The main goal of this paper is to present a criterion for determining whether the reflection φ is separating in terms of equations in the rational group algebra $\mathbb{Q}[G]$ of G. Our first main result is Theorem 11 and its Corollary 12 given in section 4. The criterion is easiest to apply in the case that the rotation group G is abelian yielding our second main result Theorem 16. Some abelian examples are given in section 5 to illustrate the theorem.

We shall assume that the tiles are all triangles. Though this imposes a fairly stringent restriction on the surfaces S, it is precisely these surfaces that are interesting because the tiling group is large in comparison to the genus (see [8]). In fact, by the Riemann-Hurwitz formula (equation 3 below), we have:

$$|G^*| > 4(\sigma - 1).$$

Furthermore, the tiling is easy to work with in this situation.

Besides the main results given in section 4 and examples given in section 5, the other main parts of the paper are as follows. In section 2 we introduce the relevant background on tilings, in particular defining the conformal tiling group G. In section 3 we use the tiling to construct the homology groups $H_q(S;\mathcal{R})$, for an arbitrary ring $\mathcal{R}$, as explicit $\mathcal{R}[G]$ modules . This explicit construction of the homology groups is then used in section 4 to develop the separability criteria in terms of the group algebra.

Though we only consider abelian examples we shall pursue a general development up until section 5. The method will be extended to non-abelian examples in a future work.

A number of authors have considered reflections on Riemann surfaces and their associated tiling groups. Two early papers are [8] and [5]. The connection of the tilings to regular maps on surfaces is explored in [8]. In the more recent papers, [3] and [4], infinite families of surfaces with conformal tiling groups of the same type are explored. The methods of determining the separability of reflections in these papers is quite different from our methods. The bibliographies of these works cite numerous other references on separability of symmetries

Acknowledgments Some of the examples were verified by other means by Baeth, Deblois and Powell [6] working at the Rose-Hulman NSF-REU, during the summer 1999 supported by NSF grant #DMS-9619714. Their work will appear in a forthcoming technical report [6], and is based on the work of Belk [1] who worked on the same problem area in the summer of 1997.

2 The kaleidoscopic tiling defined by mirrors

The tiling defined by the mirrors Remove from S all the mirrors S_ψ, as ψ varies over all reflections of the surface S. We shall assume that the components of this complement are the interiors of triangles, with respect to the hyperbolic geometry on S. Since $\sigma \geq 2$ then $\mathrm{Aut}^*(S)$ is finite and there are only a finite number of triangles. The closures of the components shall be called tiles, they meet either along edges or in vertices. In Figure 1 we have shown the tiling corresponding to the icosahedral group of isometries of the sphere. The mirrors are all great circles. Though the sphere is not hyperbolic it illustrates all the key properties of the tiling.

Pick a distinguished tile Δ_0, called the master tile, we shall assume that Δ_0 meets S_φ in a single edge. In Figure 2 we have drawn a picture of the master tile with curved edges, reminding us that it is a hyperbolic triangle. The bottom edge is the intersection $\Delta_0 \cap S_\varphi$. By construction any two tiles meeting along an edge e are interchanged by the reflection r_e in their common edge. In Figure 2 the reflected image, $\varphi\Delta_0$, in the bottom edge of the master tile, has also been drawn.

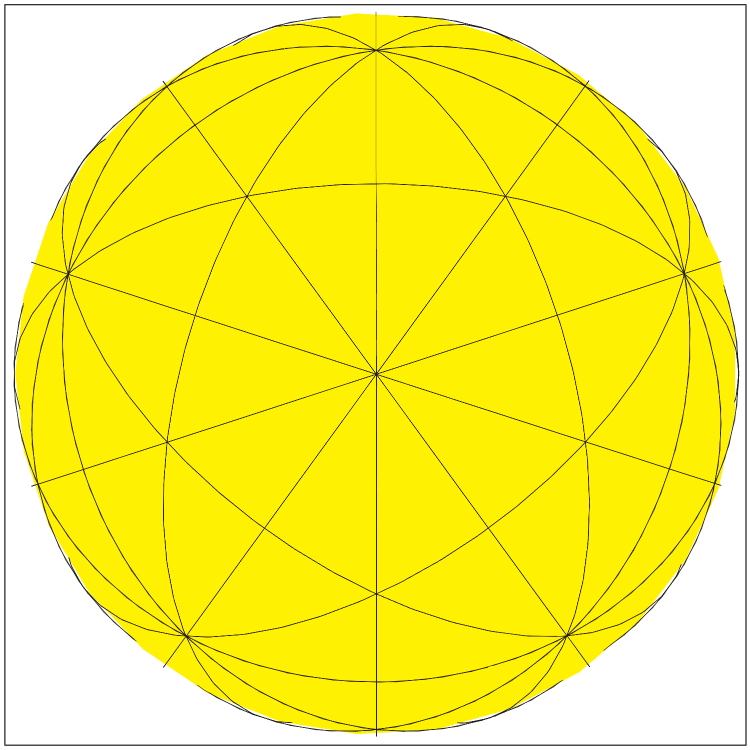

Figure 1. Icosahedral tiling of the sphere.

Let the vertices of the master tile be labelled P, Q, R as in Figure 2, and let the opposite edges be denoted by $p, q,$ and r. For the homology calculations in section 3 let us orient the master tile so that the induced orientation on the boundary is counterclockwise, i.e., we meet the vertices in the order P, Q, R as we move around the boundary in the positive direction. This also induces an orientation on the edges, i.e., $p = \overline{QR}$, $q = \overline{RP}$, $r = \overline{PQ}$ in standard, oriented line segment notation.

For simplicity of notation we let $p, q,$ and r also denote the reflections $r_p, r_{q,}$ and r_r in these edges. We have chosen the tile labelling so that $q = \varphi$. If using the common notation p, q, and r for edges and reflections causes confusion we will denote the edges by e_p, e_q, and e_r. The vertices, edges and their φ - images are shown in Figure 2.

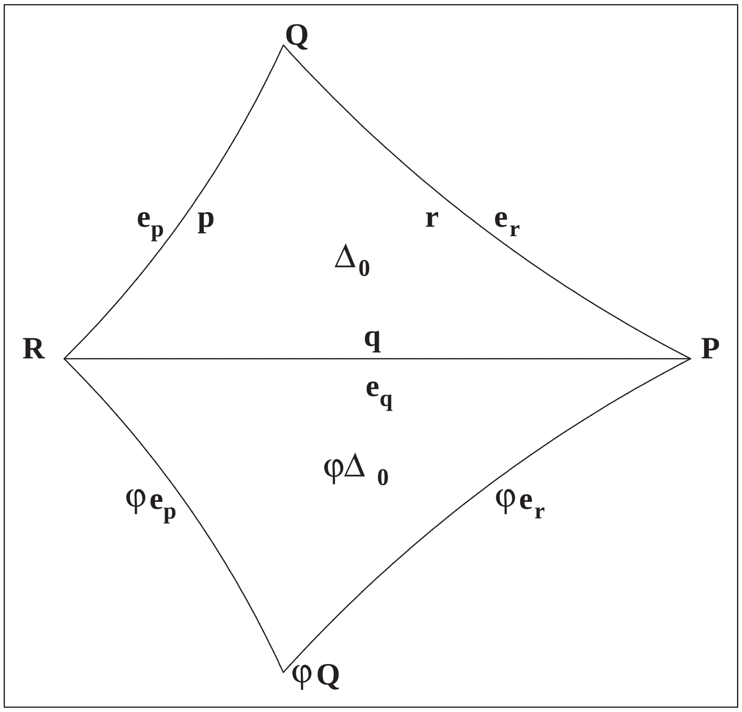

Figure 2. The master tile and its φ - image.

The tiling groups Now let $\Delta_0 = \Delta PQR$ be any triangle on a surface S with sides p, q and r as pictured in Figure 3. Assume further that the local reflections in the sides of Δ_0 each extend to global reflections of the surface. The reflected images, $p\Delta_0$, $q\Delta_0$, and $r\Delta_0$, of Δ_0 in the sides p, q, and r have been drawn with dotted lines in Figure 3. The reflections generate a finite group G^*, and as we shall see below the images of Δ_0 under G form a tiling of S. We think of Δ as a hyperbolic kaleidoscope, hence the name kaleidoscopic tiling. Since the stabilizer at a vertex of Δ_0 is a dihedral group, generated by two of the reflections in $\{p, q, r\}$, then each vertex has an even number of triangles surrounding it. It follows that the angles at R, P and Q must be $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$ for integers l, m, n, as illustrated in Figure 3. We call Δ_0 an (l, m, n)-triangle. The tilings constructed above from all reflections on the surface are examples of such tilings, but there are other examples. See Remark 1 below.

Define the elements

$$a = pq,\ b = qr,\ c = rp.$$

As Δ_0 is an (l, m, n) - triangle then $a = pq$ is easily seen to be a counter-clockwise non-euclidean rotation through $\frac{2\pi}{l}$ radians, mapping $q\Delta_0$ onto $p\Delta_0$ in Figure 3. Similarly, $b = qr$ and $c = rp$ are counter-clockwise rotations through $\frac{2\pi}{m}$ radians and $\frac{2\pi}{n}$ radians respectively. From these observations and the fact that reflections have order 2, we get the following:

$$o(a) = l,\ o(b) = m,\ o(c) = n, \tag{1}$$

and

$$abc = 1, \tag{2}$$

since $pqqrrp = 1$.

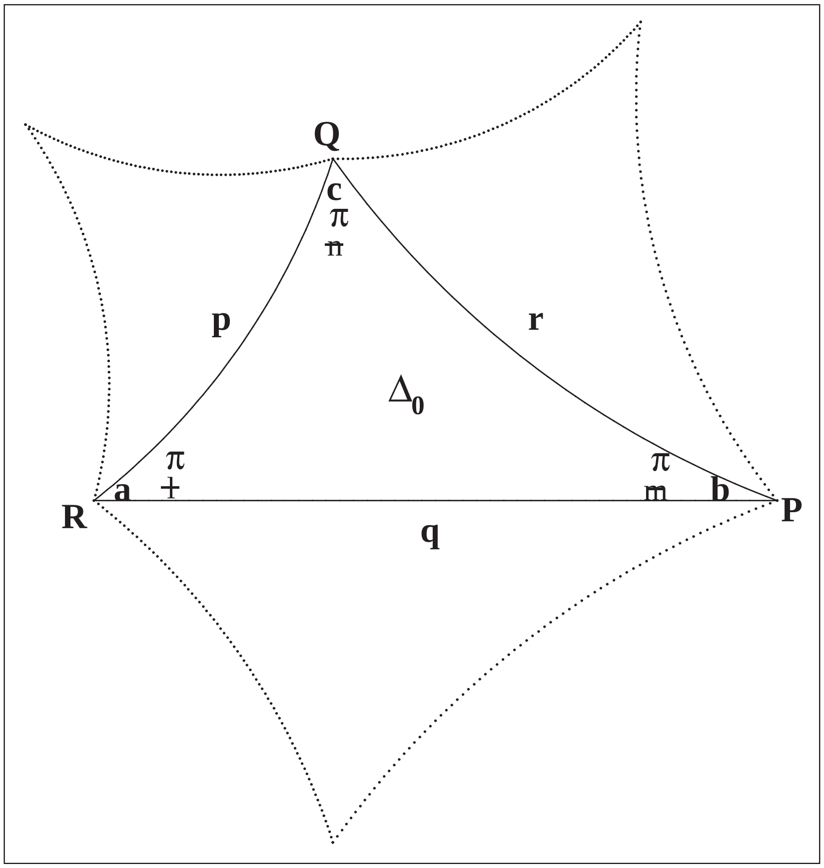

Figure 3. The master tile, reflected images, and group generators.

Let $G^* = \langle p, q, r \rangle$ and $G = \langle a, b, c \rangle = \langle a, b \rangle$ be the groups generated by the above elements. Note that G is the subgroup of orientation preserving isometries in G^*, and G is normal in G^* of index 2, in fact $G^* = \langle q \rangle \ltimes G$, a semi-direct product. The set $\{g\Delta_0 : g \in G^*\}$ forms a tiling of S by (l, m, n) - triangles.

Definition 1 *The group G^* defined above will be called the* (full) tiling group *of S (with its given tiling). The subgroup G will be called the* conformal tiling group.

Remark 1 *There may be isometries of S that preserve the tiling but are not contained in G^*. In this case the total symmetry group of the tiling has a factorization UG^*, where U is the stabilizer of the master tile. Since U is a group of isometries of a tile it is a subgroup of Σ_3, the symmetric group on the vertices of the master tile. Since the term "symmetry group" of an object usually refers to the totality of self-transformations of the object preserving the given structure the use of the term symmetry group to describe G^* would be misleading. Therefore, we use the term tiling group. Since the tilings constructed from S_φ contain all reflections in $Aut^*(S)$ then $G^* = Aut^*(S)$ in this case.*

The relation between the group order G and the genus σ of the surface is given by the Riemann-Hurwitz equation:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right) \tag{3}$$

or

$$\sigma = \frac{|G|}{2}\left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right) + 1 \tag{4}$$

The conjugation action of q on the generators a, b of G induces an automorphism of G, denoted by θ, satisfying:

$$\theta(a) = qaq = qp = a^{-1}, \tag{5}$$

$$\theta(b) = qbq = rq = b^{-1} \tag{6}$$

The triple of elements (a, b, c) of elements from G which generates G and satisfies (1) and (2) is a called a *generating* (l, m, n) - *triple* of G. Given a tiling we get a group G, a generating (l, m, n) - triple (a, b, c) and the automorphism θ. The converse is given by the following theorem, which may be found in [8]. First some nomenclature. Just as we call G^* a *reflection group* since it is generated by reflections we will call a *rotation group* any group G generated by rotations of a surface S, i.e., elements with isolated fixed points on S.

Theorem 2 *Let the G have a generating (l, m, n)-triple and suppose that the quantity σ defined by (4) is an integer. Then there is always a surfaces S of genus σ with an orientation preserving G-action, thus G is a rotation group of S. If, in addition, there is an involutary automorphism θ of G satisfying*

(5) and (6), then the surface S has a tiling by (l,m,n)-triangles such that orientation preserving tiling group as constructed above is the original G, and such that $G^* \simeq \langle\theta\rangle \ltimes G$.

Stabilizers and labelling. We may use the elements of the group to label the tiles, edges and vertices of the tiling. Since S is connected, it is fairly easy to show that G^* acts transitively on the tiles. In fact more is true.

Proposition 3 *Let Δ_0 be an (l,m,n) - triangle generating a tiling of a surface S, let G^* and G be the corresponding tiling groups, and let all other notation be as above. Then G^* acts simply transitively on the tiles, or equivalently, every tile is G - equivalent to exactly one Δ_0 or $q\Delta_0$. Likewise every edge and vertex of the equivalent to a unique edge or vertex of Δ_0. More precisely, in terms of stabilizers:*

$$\begin{aligned}
\mathrm{Stab}_{G^*}(\Delta_0) &= \{1\}, \\
\mathrm{Stab}_{G^*}(e_p) &= \{1,p\},\ \mathrm{Stab}_{G^*}(e_q) = \{1,q\},\ \mathrm{Stab}_{G^*}(e_r) = \{1,r\}, \\
\mathrm{Stab}_{G^*}(R) &= \langle p,q\rangle\,,\ \mathrm{Stab}_{G^*}(P) = \langle q,r\rangle\,,\ \mathrm{Stab}_{G^*}(Q) = \langle r,p\rangle\,, \\
\mathrm{Stab}_{G}(R) &= \langle a\rangle\,,\ \mathrm{Stab}_{G}(P) = \langle b\rangle\,,\ \mathrm{Stab}_{G}(Q) = \langle c\rangle\,.
\end{aligned}$$

Thus we may label the tiles uniquely as $g\Delta_0$ or $gq\Delta_0$, $g \in G$ and the edges uniquely by ge_p, ge_q. or ge_r, $g \in G$. Vertices cannot be uniquely labelled.

Conjugacy of reflections Since a reflection $\psi \in G^*$ is orientation-reversing and involutary, it must have the form qg, $g \in G$, where g satisfies $1 = qgqg$ or $\theta(g) = g^{-1}$. Two such reflections qg and qh are conjugate if and only if there is an $x \in G$ such that $g = \theta(x)hx^{-1}$. By simple transitivity, a reflection $\psi \in G^*$ cannot fix an interior point of a tile so that it must be the reflection r_e where e is some edge in the tiling. Since each edge is G - equivalent to one of the edges of the master tile, and since a reflection in an edge satisfies $r_{ge} = gr_eg^{-1}$; it follows that ψ is conjugate to one of p, q, or r. In particular φ is conjugate to p if an only if

$$q = \varphi = gpg^{-1} = gqa^{-1}g^{-1} = q\theta(g)a^{-1}g^{-1},$$

or

$$a^{-1} = \theta(g^{-1})g, \text{ for some } g \in G. \tag{7}$$

Similarly, φ is conjugate to r if and only if

$$b = \theta(h^{-1})h, \text{ for some } h \in G. \tag{8}$$

Note that setting $g = a^s$ in 7 leads to solving $a^{-1} = a^{2s}$ or $a^{2s+1} = 1$ for some s. This can be solved for s if and only if a has odd order. Thus if a is odd then p and q are automatically conjugate. Similarly q and r are automatically conjugate if b has odd order, and p and r are automatically conjugate if c has odd order.

The induced tiling on the mirrors For our homology calculations later we will need to know the tiling of the ovals by edges, induced by the tiling of the surface. Since G acts freely on the edges, we may color each of the edges of the tiling, and give it a direction, according to which edge p, q, or r the given edge is equivalent to. As we move around an oval we will traverse the edges in a certain color and direction pattern. For instance, if we let the colours be p, q, and r, with the obvious interpretation, then the pattern $p^+q^+r^+r^-q^-p^-$ means we traverse three edges coloured p, q, r in the positive direction and then three edges coloured r, q, p in the negative direction. As we move along the oval the pattern keeps cyclically repeating itself. For example in the icosahedral tiling on the sphere in Figure 1 every oval has twelve edges with colour pattern $q^+r^+p^+p^-r^-q^-q^+r^+p^+p^-r^-q^-$. The pattern may depend on which direction we traverse the oval, so each will have one or two patterns (up to cyclic permutation). By definition, two ovals will be G - equivalent if and only if they have a common pattern. Furthermore, every oval is equivalent to one of the three ovals $\mathcal{O}_p$, $\mathcal{O}_q$, or $\mathcal{O}_r$, - the ovals containing p, q, and r, respectively - so we need only determine the patterns of these three ovals.

We may determine the various color and direction patterns by constructing the *dissected boundary,* $\partial^d(\Delta_0)$, of the master tile. Take the boundary of the master tile and cut it at all the vertices of even order, and then take the closure of the components. The resulting set $\partial^d(\Delta_0)$ consists of a either a circle or one, two or three segments. The different color and direction pattern of ovals are determined by the elements of the dissected boundary. We have eight possible cases, depending on the parity of l, m, and n. In Table 1 below we give the colour pattern of the three basic ovals in the eight cases. In the table the parity descriptor EOO , for example, indicates that l is even and

m and n are odd.

Table 1. Colour patterns of ovals

parity	$\mathcal{O}_p$	$\mathcal{O}_q$	$\mathcal{O}_r$
OOO	$p^+q^+r^+, r^-q^-p^-$	$p^+q^+r^+, r^-q^-p^-$	$p^+q^+r^+, r^-q^-p^-$
EOO	$q^+r^+p^+p^-r^-q^-$	$q^+r^+p^+p^-r^-q^-$	$q^+r^+p^+p^-r^-q^-$
OEO	$r^+p^+q^+q^-p^-r^-$	$r^+p^+q^+q^-p^-r^-$	$r^+p^+q^+q^-p^-r^-$
OOE	$p^+q^+r^+r^-q^-p^-$	$p^+q^+r^+r^-q^-p^-$	$p^+q^+r^+r^-q^-p^-$
OEE	$p^+q^+q^-p^-$	$p^+q^+q^-p^-$	r^+r^-
EOE	p^+p^-	$q^+r^+r^-q^-$	$q^+r^+r^-q^-$
EEO	$r^+p^+p^-r^-$	q^+q^-	$r^+p^+p^-r^-$
EEE	p^+p^-	q^+q^-	r^+r^-

Let $\mathcal{O}$ be an oval of the reflection ψ. There may be more than one type of oval pattern in the mirror of ψ so for each component B of the dissected boundary we let $\mathcal{O}(\psi, B)$ denote the set of ovals of ψ that correspond to B. The following facts are proven in [2].

Proposition 4 *Let all notation be as above then:*

1. $\mathrm{Cent}_{G^*}(\psi)$ *permutes* $\mathcal{O}(\psi, B)$ *transitively.*
2. $\mathrm{Cent}_{G^*}(\psi) = \langle\psi\rangle \times \mathrm{Cent}_G(\psi)$.
3. *If* $\mathcal{O}$ *is an oval in* $\mathcal{O}(\psi, B)$*, then the number of ovals is given by:*

$$|\mathcal{O}(\psi, B)| = \frac{|\mathrm{Cent}_G(\psi)|}{|\mathrm{Stab}_G(\mathcal{O})|}.$$

4. *If* $\mathcal{O}$ *is an oval in* $\mathcal{O}(\psi, B)$*, then there are homeomorphisms:*

$$\mathcal{O}(\psi, B)/\mathrm{Cent}_G(\psi) \simeq \mathcal{O}/\mathrm{Stab}_G(\mathcal{O}) \simeq B$$

5. *If* B *is a circle then the stabilizer of the oval is cyclic*

$$\mathrm{Stab}_G(\mathcal{O}) \simeq \mathbb{Z}_M,$$

and if B *is a segment then stabilizer of the oval is dihedral.*

$$\mathrm{Stab}_G(\mathcal{O}) \simeq D_M.$$

The stabilizers in the various cases can be explicitly computed. First a bit of notation. Let h_p, h_q, and h_r denote a generator of the rotational subgroup of $\mathrm{Stab}_G(\mathcal{O}_p)$, $\mathrm{Stab}_G(\mathcal{O}_q)$, and $\mathrm{Stab}_G(\mathcal{O}_r)$, respectively. Let λ, μ, ν be integers such that

$$l = 2\lambda,\ m = 2\mu,\ n = 2\nu,$$

in the even cases and

$$l = 2\lambda + 1,\ m = 2\mu + 1,\ n = 2\nu + 1,$$

in the odd cases. The oval stabilizers presented in Tables 2 and 3 below are easily calculated from the results in [2] or [7].

Table 2. Stabilizers of ovals

parity	$\mathrm{Stab}_G(\mathcal{O}_p)$	$\mathrm{Stab}_G(\mathcal{O}_q)$	$\mathrm{Stab}_G(\mathcal{O}_r)$
OOO	$\langle a^{\lambda+1}b^{\mu+1}c^{\nu+1}\rangle$	$\langle b^{\mu+1}c^{\nu+1}a^{\lambda+1}\rangle$	$\langle c^{\nu+1}a^{\lambda+1}b^{\mu+1}\rangle$
EOO	$\langle c^{\nu}b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu+1}, a^{\lambda}\rangle$	$\langle a^{\lambda}, b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}b^{\mu}\rangle$	$\langle b^{\mu}a^{\lambda}b^{\mu+1}, c^{\nu+1}a^{\lambda}c^{\nu}\rangle$
OEO	$\langle c^{\nu}b^{\mu}c^{\nu+1}, a^{\lambda+1}b^{\mu}a^{\lambda}\rangle$	$\langle a^{\lambda}c^{\nu}b^{\mu}c^{\nu+1}a^{\lambda+1}, b^{\mu}\rangle$	$\langle b^{\mu}, c^{\nu+1}a^{\lambda+1}b^{\mu}a^{\lambda}c^{\nu}\rangle$
OOE	$\langle c^{\nu}, a^{\lambda+1}b^{\mu+1}c^{\nu}b^{\mu}a^{\lambda}\rangle$	$\langle a^{\lambda}c^{\nu}a^{\lambda+1}, b^{\mu+1}c^{\nu}b^{\mu}\rangle$	$\langle b^{\mu}a^{\lambda}c^{\nu}a^{\lambda+1}b^{\mu+1}, c^{\nu}\rangle$
OEE	$\langle c^{\nu}, a^{\lambda+1}b^{\mu}a^{\lambda}\rangle$	$\langle a^{\lambda}c^{\nu}a^{\lambda+1}, b^{\mu}\rangle$	$\langle b^{\mu}, c^{\nu}\rangle$
EOE	$\langle c^{\nu}, a^{\lambda}\rangle$	$\langle a^{\lambda}, b^{\mu+1}c^{\nu}b^{\mu}\rangle$	$\langle b^{\mu}a^{\lambda}b^{\mu+1}, c^{\nu}\rangle$
EEO	$\langle c^{\nu}b^{\mu}c^{\nu+1}, a^{\lambda}\rangle$	$\langle a^{\lambda}, b^{\mu}\rangle$	$\langle b^{\mu}, c^{\nu+1}a^{\lambda}c^{\nu}\rangle$
EEE	$\langle c^{\nu}, a^{\lambda}\rangle$	$\langle a^{\lambda}, b^{\mu}\rangle$	$\langle b^{\mu}, c^{\nu}\rangle$

Table 3. Generators of rotational stabilizers of ovals

parity	h_p	h_q	h_r
OOO	$a^{\lambda+1}b^{\mu+1}c^{\nu+1}$	$b^{\mu+1}c^{\nu+1}a^{\lambda+1}$	$c^{\nu+1}a^{\lambda+1}b^{\mu+1}$
EOO	$c^{\nu}b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}$	$a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}b^{\mu}$	$b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}$
OEO	$c^{\nu}b^{\mu}c^{\nu+1}a^{\lambda+1}b^{\mu}a^{\lambda}$	$a^{\lambda}c^{\nu}b^{\mu}c^{\nu+1}a^{\lambda+1}b^{\mu}$	$b^{\mu}c^{\nu+1}a^{\lambda+1}b^{\mu}a^{\lambda}c^{\nu}$
OOE	$c^{\nu}a^{\lambda+1}b^{\mu+1}c^{\nu}b^{\mu}a^{\lambda}$	$a^{\lambda}c^{\nu}a^{\lambda+1}b^{\mu+1}c^{\nu}b^{\mu}$	$b^{\mu}a^{\lambda}c^{\nu}a^{\lambda+1}b^{\mu+1}c^{\nu}$
OEE	$c^{\nu}a^{\lambda+1}b^{\mu}a^{\lambda}$	$a^{\lambda}c^{\nu}a^{\lambda+1}b^{\mu}$	$b^{\mu}c^{\nu}$
EOE	$c^{\nu}a^{\lambda}$	$a^{\lambda}b^{\mu+1}c^{\nu}b^{\mu}$	$b^{\mu}a^{\lambda}b^{\mu+1}c^{\nu}$
EEO	$c^{\nu}b^{\mu}c^{\nu+1}a^{\lambda}$	$a^{\lambda}b^{\mu}$	$b^{\mu}c^{\nu+1}a^{\lambda}c^{\nu}$
EEE	$c^{\nu}a^{\lambda}$	$a^{\lambda}b^{\mu}$	$b^{\mu}c^{\nu}$

Finally we relate the stabilizers to the colour patterns.

Proposition 5 *Let all notation be as above. Then, fundamental regions for the rotational stabilizer of ovals are the colour patterns given in Table 1. If* *in addition, the stabilizer of an oval is dihedral that then we may take the positive part of the colour pattern as a fundamental region of the dihedral action.*

3 The homology of S as G-modules

We may use the tiling of the surface to define a cell complex and the homology groups of S. The natural action of G allows us then to explicitly write the homology as G - modules. Let $\mathcal{R}$ be an arbitrary ring with identity and let $C_2(S), C_1(S)$, and $C_0(S)$ be the free $\mathcal{R}$-modules whose bases are the triangles, the edges and the vertices of the tiling. Let $\mathcal{A} = \mathcal{R}[G]$ be the $\mathcal{R}$ - group algebra of G. Because the G - action on S preserves the tiling, then these cell complexes are $\mathcal{A}$ - modules. In fact we have:

$$\begin{aligned} C_2(S) &= \mathcal{A}\Delta_0 \oplus \mathcal{A}q\Delta_0, \\ C_1(S) &= \mathcal{A}e_p \oplus \mathcal{A}e_q \oplus \mathcal{A}e_r, \\ C_0(S) &= \mathcal{A}P \oplus \mathcal{A}Q \oplus \mathcal{A}R. \end{aligned}$$

All the summands of these modules are free except for the summands of $C_0(S)$.

Remark 2 *There is an advantage to letting $\mathcal{R}$ be arbitrary, at least an arbitrary field, even though the results are independent of characteristic. For instance compare the simplification achieved in Remark 4 and Example 2, section 5 with $\mathcal{R} = \mathbb{F}_2$ and the use of $1 \neq -1$ in the calculations in Example 1, section 5, with $\mathcal{R} = \mathbb{Q}$.*

From Figure 2 we are easily able to calculate that $\partial(\Delta_0) = e_p + e_q + e_r$, and $\partial(q\Delta_0) = \varphi e_p + e_q + \varphi e_r = qe_p + e_q + qe_r$ since $q\Delta_0 = \varphi\Delta_0$ is considered to be the triangle $\Delta(R, P, \varphi(Q))$. But now $qe_p = qpe_p = a^{-1}e_p$ and $qe_r = qre_r = be_r$, thus we may express $\partial(q\Delta_0) = a^{-1}e_p + e_q + be_r$ with coefficients from $\mathcal{A}$ only. The boundary map is a G - module homomorphism so we get:

$$\begin{aligned} \partial(\zeta\Delta_0 + \eta q\Delta_0) &= \zeta\partial(\Delta_0) + \eta\partial(q\Delta_0) \\ &= \zeta(e_p + e_q + e_r) + \eta(a^{-1}e_p + e_q + be_r) \\ &= (\zeta + \eta a^{-1})e_p + (\zeta + \eta)e_q + (\zeta + \eta b)e_r, \end{aligned} \tag{9}$$

$$\partial(\alpha e_p + \beta e_q + \gamma e_r) = \alpha\partial e_p + \beta\partial e_q + \gamma\partial e_r \qquad (10)$$
$$= \alpha(R - Q) + \beta(P - R) + \gamma(Q - P)$$
$$= (\beta - \gamma)P + (\gamma - \alpha)Q + (\alpha - \beta)R,$$

and

$$\partial(\rho P + \sigma Q + \tau R) = \epsilon(\rho + \sigma + \tau), \qquad (11)$$

where $\epsilon : \mathcal{A} \to \mathcal{R}$ is the augmentation map that calculates the sum of the coefficients of an element in the group algebra. From these formulae we easily get the following lemma.

Lemma 6 *Let $Z_k(S)$ and $B_k(S)$ be the modules of k-cycles and k-boundaries. Then $\alpha e_p + \beta e_q + \gamma e_r \in Z_1(S)$ if and only if*

$$\beta - \gamma \in \mathcal{A}(1 - a),\ \gamma - \alpha \in \mathcal{A}(1 - b),\ \alpha - \beta \in \mathcal{A}(1 - c).$$

For boundaries, $\alpha e_p + \beta e_q + \gamma e_r \in B_1(S)$ if and only if there are $\zeta, \eta \in \mathcal{A}$ such that

$$\alpha = \zeta + \eta a^{-1},\ \beta = \zeta + \eta,\ \gamma = \zeta + \eta b, \qquad (12)$$

or

$$\eta(1 - a^{-1}) = \beta - \alpha,\ \eta(1 - b) = \beta - \gamma. \qquad (13)$$

Proof. For statement 1, if $\alpha e_p + \beta e_q + \gamma e_r$ is a cycle then $\beta - \gamma$ is in the annihilator of P. This is the left ideal $\mathcal{A}(1 - a)$. The remaining details for statement 1 are similar. For statement 2, first observe that if we set $\zeta = \beta - \eta$ then the systems of equations in 12 and 13 are easily seen to be equivalent. Statement 2 now follows from equation 9 above.

We finish this section with an easily proved lemma on boundaries of certain 2-chains.

Lemma 7 *Let $H \subseteq G^*$ be any subset. Let $sgn : G^* \to \{1, -1\}$ be the sign character that equals 1 on G and -1 on $G^* - G$. Let $c(H)$ be the following chain*

$$c(H) = \sum_{g \in H} sgn(g) g \Delta_0.$$

Then, the support of $\partial(c(H))$ is the set of all edges that belong to exactly one $\{g\Delta_0 : g \in H\}$. Alternatively ge_p is in the support of $\partial(c(H))$ if and only if exactly one of g and gp is in H. Moreover, the coefficient of any term in $\partial(c(H))$ is ± 1. Similar statements hold for q and r.

Proof. The alternative statement for ge_p follows from the fact that $g\Delta_0$ and $gp\Delta_0$ are the only triangles that meet along ge_p. For the coefficient c of ge_p in $\partial(c(H))$ is given by

$$c = sgn(g) + sgn(gp) = 0, \text{ if } \{g, gp\} \cap H = \{g, gp\}, \tag{14}$$
$$c = sgn(g) \text{ or } sgn(gp) = \pm 1, \text{ if } \{g, gp\} \cap H = \{g\} \text{ or } \{gp\}, \tag{15}$$
$$c = 0, \text{if } \{g, gp\} \cap H = \phi. \tag{16}$$

Similar proofs hold for q and r. ■

Remark 3 *If $\mathcal{R}$ is a field with characteristic 2 then all of the results of this section are valid provided that -1 is interpreted as 1. The interpretation of the results are significantly different in this case. This also applies to the development in the next section.*

4 A separability criterion

The following proposition is the starting point for using the tiling and the group algebra to determine separability. If φ separates S then the closure of each half is an orientable surface with boundary S_φ. Thus S_φ is homologous to zero in S. This is formalized by the following proposition. In this proposition and the rest of this section we will assume that $\mathcal{R}$ is a field.

Proposition 8 *Let φ be a reflection of the Riemann surface S. Then, φ is separating if and only the inclusion map in homology.*

$$\iota : H_1(S_\varphi) \longrightarrow H_1(S) \tag{17}$$

has non-trivial kernel.

Proof. Since the homology group $H_2(S_\varphi)$ is trivial then the long exact sequence in homology gives:

$$H_2(S) \hookrightarrow H_2(S, S_\varphi) \longrightarrow H_1(S_\varphi) \xrightarrow{\iota} H_1(S).$$

Now $H_2(S) \simeq \mathcal{R}$, where $\mathcal{R}$ is the coefficient ring. Also, by the universal coefficient theorem for cohomology for fields and Poincaré duality we have

$$H_2(S, S_\varphi) \simeq H^2(S, S_\varphi) \simeq H_0(S - S_\varphi) \simeq \mathcal{R}^k,$$

where k is the number of components of $S - S_\varphi$ (see p. 244 and p. 267 of [9]). Thus, the map ι is injective if and only if φ is not separating. ■

Now we assume for the rest of this section that φ is separating. We need identify the kernel of the map in equation 17. Let $S^+$and S^- denote the closures of the two components of $S - S_\varphi$, assuming that $\Delta_0 \subset S^+$. By construction, S^+ is a union of tiles so we may write

$$S^+ = \bigcup_{g \in G^+} g\Delta_0.$$

for a suitably chosen subset $G^+ \subset G^*$. Let $c(G^+)$ denote the chain defined by:

$$c(G^+) = \sum_{g \in G^+} sgn(g) g\Delta_0$$

as in Lemma 7. When the boundary $\partial(c(G^+))$ is computed all the edges cancel except those that lie on S_φ, according to Lemma 7. Thus $\partial(c(G^+))$ is 1-boundary whose support is all of S_φ. Since $\partial(c(G^+))$ is a non-zero 1-cycle of S_φ that is a boundary in S then it lies in the kernel of the map in equation 17.

Now let us find a representation of $\partial(c(G^+))$ as an element of $\mathcal{A}e_p \oplus \mathcal{A}e_q \oplus \mathcal{A}e_r$. As S_φ may have several types of ovals, we need to find a separate expression for each type. Let H_p denote $\mathrm{Cent}_G(p)$ and let H_q and H_r be similarly defined. Since every element h of H_q maps S_φ to itself, then h maps both of S^+ and S^- to themselves or switches them. Define a map $sgn_q : H_q \to \{1, -1\}$ such that

$$sgn_q(h) = 1 \text{ iff } hS^+ = S^+,$$
$$sgn_q(h) = -1 \text{ iff } hS^+ = S^-.$$

Observe that sgn_q is different from the orientation character sgn defined on all G^*, and that h restricted to S_φ is orientation preserving if and only if $sgn_q(h) = 1$. Now select a sequence of positively oriented edges on $\mathcal{O}_q$ containing e_q, with the aid of Table 1. This sequence is a fundamental region for the action of H_q on $\mathcal{O}(\varphi, B_q)$ where B_q is the component of the dissected boundary $\partial^d(\Delta_0)$ containing e_q. Let x_q be the element in $C_1(S)$ obtained as the sum of these edges as listed in Table 4.a. Then the contribution to $\partial(c(G^+))$ from $\mathcal{O}_q$ is:

$$\mathcal{M}_q = \sum_{h \in H_q} sgn_q(h) h x_q \tag{18}$$

The support of the chain $\mathcal{M}_q$ is the union of the ovals of S_φ conjugate to $\mathcal{O}_q$. The quantity $\mathcal{M}_q$ is all we need in the first four cases of Table 1. In the last four cases we may need additional quantities coming from p and r. We may similarly define $\mathcal{M}_p$ and $\mathcal{M}_r$ and x_p and x_r if p or r, respectively is separating.

Table 4.a The elements x_q

parity	x_q
OOO	$a^\lambda e_p + e_q + b^{\mu+1} e_r$
EOO	$b^{\mu+1} c^{\nu+1} e_p + e_q + b^{\mu+1} e_r$
OEO	$a^\lambda e_p + e_q + a^\lambda c^\nu e_r$
OOE	$a^\lambda e_p + e_q + b^{\mu+1} e_r$
OEE	$a^\lambda e_p + e_q$
EOE	$e_q + b^{\mu+1} e_r$
EEO	e_q
EEE	e_q

Table 4.b The elements x_p and x_r

parity	x_p	x_r
OOO	$e_p + a^{\lambda+1} e_q + c^\nu e_r$	$c^{\nu+1} e_p + b^\mu e_q + e_r$
EOO	$e_p + c^\nu b^\mu e_q + c^\nu e_r$	$c^{\nu+1} e_p + b^\mu e_q + e_r$
OEO	$e_p + a^{\lambda+1} e_q + c^\nu e_r$	$c^{\nu+1} e_p + c^{\nu+1} a^{\lambda+1} e_q + e_r$
OOE	$e_p + a^{\lambda+1} e_q + a^{\lambda+1} b^{\mu+1} e_r$	$b^\mu a^\lambda e_p + b^\mu e_q + e_r$
OEE	$e_p + a^{\lambda+1} e_q$	e_r
EOE	e_p	$b^\mu e_q + e_r$
EEO	$e_p + c^\nu e_r$	$c^{\nu+1} e_p + e_r$
EEE	e_p	e_r

The foregoing may be summarized in the following proposition.

Proposition 9 *Suppose that φ is a separating reflection and $\mathcal{R}$ is a field. Then a cycle spanning the kernel of the homology map $H_1(S_\varphi) \longrightarrow H_1(S)$ is given in the following table.*

Table 5. *Spanning Cycles*

Spanning cycle	*Cases*
$\mathcal{M}_q$	*all cases, except as in the lines below and in* *
$\varepsilon_g g \mathcal{M}_p + \mathcal{M}_q$	EOE, EEO, EEE *and* $a^{-1} = \theta(g^{-1})g$ **
$\mathcal{M}_q + \varepsilon_h h \mathcal{M}_r$	OEE, EEE *and* $b = \theta(h^{-1})h$ ***
$\varepsilon_g g \mathcal{M}_p + \mathcal{M}_q + \varepsilon_h h \mathcal{M}_r$	EEE *and* $a^{-1} = \theta(g^{-1})g$ *and* $b = \theta(h^{-1})h$

* *but* φ *not conjugate to* p *or* r *unless forced by parity considerations*
** *in case EEE,* φ *is not conjugate to* r
*** *in case EEE,* φ *is not conjugate to* p

The numbers ε_g *and* ε_h *are* $+1$ *or* -1, *depending whether* g, *respectively* h, *preserve orientation when mapping* $\mathcal{O}_p$, *respectively* $\mathcal{O}_r$, *into* S_φ.

Proof. We just prove the first case, the remaining ones are similar. Since $\mathcal{R}$ is a field then the kernel of $H_1(S_\varphi) \longrightarrow H_1(S)$ is spanned by a single cycle, which we may take to be $\partial(c(G^+))$. By Proposition 5 every edge e in the support of $\partial(c(G^+))$ is H_q - equivalent to a unique edge e' in the 1-chain x_q, i.e., $e = he'$ for some $h \in H_q$. Furthermore, e' is the positively oriented intersection of S_φ with a tile of the form $k\Delta_0 \subset S^+$ where $k \in G$. This is most easily verified by looking at the table though, also, it can be proven by parity considerations. Now suppose that $sgn_q(h) = 1$, then $hk\Delta_0 \subset S^+$. Thus the only edge coming from $\partial(hk\Delta_0)$ surviving cancellation in the sum $\partial(c(G^+)) = \sum_{g\in G^+} sgn(g)\partial(g\Delta_0)$ is $sgn(hk)e = sgn_q(h)he'$. If $sgn_q(h) = -1$ then $\varphi hk\Delta_0 \subset S^+$ and the corresponding term in $\partial(c(G^+))$ is $sgn(\varphi hk)e = sgn_q(h)he'$. ■

Before formulating our separability criteria let us record some easily proven properties of the function $sgn_q : H_q \to \{1, -1\}$ and an alternative look at $\mathcal{M}_q$

Proposition 10 *Let* φ *be a separating reflection of* S, *let* $H_q = \mathrm{Cent}_G(\varphi)$ *and let* $sgn_q : H_q \to \{1, -1\}$ *be as defined above. Let* h_q *be the generator of the rotational part of* $\mathrm{Stab}_G(\mathcal{O}_q)$. *Then,*

$$sgn_q(h_q) = 1. \tag{19}$$

Further suppose that one of l, m, n *is even, and let* $z \in H_q$ *be any involution fixing a vertex of even order on* $\mathcal{O}_q$. *Then,*

$$sgn_q(z) = -1 \tag{20}$$

Remark 4 *Let* S_q *be the stabilizer of* $\mathcal{O}_q$ *and set* $\mathcal{S}_q = \sum_{g\in S_q} sgn_q(g)g$. *This element can be determined without knowing the function* sgn_q, *or even knowing if* φ *is separating. For, by the proposition above* sgn_q *equals* 1 *on the rotational stabilizer and equals* -1 *on the involutions fixing the points of* *even order on $\mathcal{O}_q$. The cycle $\mathcal{S}_q x_q$ is the cycle representing $\mathcal{O}_q$. If $h_1, \ldots, h_k$ are right coset representatives for G/S_q we have*

$$\begin{aligned}\mathcal{M}_q &= \sum_{i=1}^{k} sgn_q(h_i) h_i \mathcal{S}_q x_q \\ &= \sum_i \epsilon_i \mathcal{O}_i\end{aligned}$$

for some coefficients $\epsilon_i = \pm 1$, and ovals $\mathcal{O}_i = h_i \mathcal{S}_q x_q$. Thus instead of trying to determine sgn_q one can just try to solve $\partial(\zeta\Delta_0 + \eta\varphi\Delta_0) = \sum_i \epsilon_i \mathcal{O}_i$ for various selections of the ϵ_i. Alternatively, we can take $\mathcal{R} = \mathbb{F}_2$, the prime Galois field of characteristic 2. Then the ϵ_i all equal 1 and there is only one combination of ovals to consider.

Remark 5 *When the stabilizer S_q is dihedral $\mathcal{S}_q$ has a special factorization. In fact $S_q = \langle z \rangle \ltimes \langle h_q \rangle$ where z is any involution fixing an even vertex lying on $\mathcal{O}_q$. Then*

$$\begin{aligned}\mathcal{S}_q &= \sum_{g \in S_q} sgn_q(g) g \\ &= \sum_{s=1}^{o(h_q)} sgn_q(h_q^s) h_q^s + \sum_{s=1}^{o(h_q)} sgn_q(z h_q^s) z h_q^s \\ &= (1 - z) \sum_{s=1}^{o(h_q)} h_q^s = \left(\sum_{s=1}^{o(h_q)} h_q^s \right) (1 - z).\end{aligned}$$

Finally, we have our main theorem and its Corollary.

Theorem 11 *Let φ be a reflection on a surface S, inducing a triangular tiling on S with conformal tiling group G. Let $H_q = \mathrm{Cent}_G(\varphi)$, and let all other notation be as above. Then is φ separating if and only if there is a homomorphism $sgn_q : H_q \to \{1, -1\}$ satisfying equations 19 and 20, and elements $\zeta, \eta \in \mathcal{A}$ such that*

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \mathcal{M}_q, \tag{21}$$

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \varepsilon_g g\mathcal{M}_p + \mathcal{M}_q, \tag{22}$$

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \mathcal{M}_q + \varepsilon_h h\mathcal{M}_r, \text{ or} \tag{23}$$

$$\partial(\zeta\Delta_0 + \eta q\Delta_0) = \varepsilon_g g\mathcal{M}_p + \mathcal{M}_q + \varepsilon_h h\mathcal{M}_r, \tag{24}$$

with the right hand side chosen according to Table 5.

Proof. We have already shown that if φ is separable then the separability conditions 21 - 24 must hold. Now suppose that sgn_q with the required properties exists and the separability conditions 21 - 24 are satisfied. Then there is a non-zero kernel to the homology map given by 17 and φ must be separating. ■

The following corollary follows from the preceding theorem and equations 13 of Lemma 6.

Corollary 12 *Let all notation be as in the theorem above. Set*

$$\mathcal{H}_p = \sum_{h\in H_p} sgn_p(h)h,$$
$$\mathcal{H}_q = \sum_{h\in H_q} sgn_q(h)h,$$
$$\mathcal{H}_r = \sum_{h\in H_r} sgn_r(h)h,$$

and define $x_{pp}, \ldots, x_{rr}$ *by*

$$x_p = x_{pp}e_p + x_{pq}e_q + x_{pr}e_r,$$
$$x_q = x_{qp}e_p + x_{qq}e_q + x_{qr}e_r,$$
$$x_r = x_{rp}e_p + x_{rq}e_q + x_{rr}e_r.$$

Then the separability conditions 21 - 24 are equivalent to the following conditions:

$$\begin{aligned}\eta(1-a^{-1}) &= \mathcal{H}_q(x_{qq}-x_{qp}),\\ \eta(1-b) &= \mathcal{H}_q(x_{qq}-x_{qr}),\end{aligned} \tag{25}$$

$$\begin{aligned}\eta(1-a^{-1}) &= \varepsilon_g g\mathcal{H}_p(x_{pq}-x_{pp}) + \mathcal{H}_q(x_{qq}-x_{qp}),\\ \eta(1-b) &= \varepsilon_g g\mathcal{H}_p(x_{pq}-x_{pr}) + \mathcal{H}_q(x_{qq}-x_{qr}),\end{aligned} \tag{26}$$

$$\begin{aligned}\eta(1-a^{-1}) &= \mathcal{H}_q(x_{qq}-x_{qp}) + \varepsilon_h h\mathcal{H}_r(x_{rq}-x_{rp}),\\ \eta(1-b) &= \mathcal{H}_q(x_{qq}-x_{qr}) + \varepsilon_h h\mathcal{H}_r(x_{rq}-x_{rr}),\end{aligned} \tag{27}$$

$$\eta(1-a^{-1}) = \varepsilon_g g\mathcal{H}_p(x_{pq}-x_{pp}) + \mathcal{H}_q(x_{qq}-x_{qp}) + \varepsilon_h h\mathcal{H}_r(x_{rq}-x_{rp}), \tag{28}$$
$$\eta(1-b) = \varepsilon_g g\mathcal{H}_p(x_{pq}-x_{pr}) + \mathcal{H}_q(x_{qq}-x_{qr}) + \varepsilon_h h\mathcal{H}_r(x_{rq}-x_{rr}). \tag{29}$$

The right hand side of the equations are chosen according to Table 5.

Remark 6 *The element* $1-a^{-1}$ *is a zero divisor in* $\mathcal{A}$*, viz.,*

$$(1-a^{-1})(1+a^{-1}+a^{-2}+\cdots+a^{-(l-1)}) = 1-a^{-l} = 0.$$

Thus the group algebra equation 25

$$\eta(1-a^{-1}) = \mathcal{H}_q(x_{qq}-x_{qp})$$

leads to the following necessary condition for separability

$$\begin{aligned}
&\mathcal{H}_q(x_{qq}-x_{qp})(1+a+a^2+\cdots+a^{l-1}) \\
&= \mathcal{H}_q(x_{qq}-x_{qp})(1+a^{-1}+a^{-2}+\cdots+a^{-(l-1)}) \\
&= \eta(1-a^{-1})(1+a^{-1}+a^{-2}+\cdots+a^{-(l-1)}) \\
&= 0.
\end{aligned}$$

The other equations lead to similar necessary conditions, e.g.,

$$\mathcal{H}_q(x_{qq}-x_{qr})(1+b+b^2+\cdots+b^{m-1}) = 0.$$

5 Abelian examples

Abelian simplifications If G is Abelian then the situation is dramatically simplified since, θ is very simple; the ovals, mirrors and stabilizers are fairly simple; and $\mathcal{A}$ is a commutative algebra in which the solving the group algebra equation is easy. Furthermore, every abelian rotation group G extends to a tiling group G^* because of the following proposition, which may also be found in [8].

Proposition 13 *The automorphism satisfies* $\theta(g) = g^{-1}$ *for all* $g \in G$ *if and only if* G *is abelian.*

Proof. The map $i : g \longrightarrow g^{-1}$ is a automorphism if and only if G is abelian. The automorphism $\theta^{-1} \circ i$ fixes a and b, and hence it fixes $G = \langle a, b \rangle$. ■

Proposition 14 *Suppose that the conformal tiling group* G *is abelian. There is at most one or two repetitions of a pattern in an oval and the number of ovals in a mirror is one or two. The number of repetitions is given as* $o(h_p)$*,*

$o(h_q)$ or $o(h_r)$, respectively. Let $|\mathcal{M}_p|$, $|\mathcal{M}_q|$, or $|\mathcal{M}_r|$ denote the number of ovals in the three types of mirrors. Then the above quantities are given as in Table 6. If G is cyclic then all the numbers in the first three columns equal 1.

Table 6. *Oval and Mirror sizes*

parity	$o(h_p)$	$o(h_q)$	$o(h_r)$	$\|\mathcal{M}_p\|$	$\|\mathcal{M}_q\|$	$\|\mathcal{M}_r\|$
OOO	1	1	1	1	1	1
EOO	1	1	1	1	1	1
OEO	1	1	1	1	1	1
OOE	1	1	1	1	1	1
OEE	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2
EOE	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2
EEO	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2
EEE	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2	1 *or* 2

Proof. We give a few sample calculations of the 48 entries of the table. In the *OOO* case we have:

$$\begin{aligned} h_p^2 &= (a^{\lambda+1}b^{\mu+1}c^{\nu+1})^2 \\ &= a^{2\lambda+1}b^{2\mu+1}c^{2\nu+1}abc \\ &= a^l b^m c^n abc = 1. \end{aligned}$$

Thus h_p is an element of order 2 in an odd order group, and hence must be the identity. In the *EOO* case

$$\begin{aligned} h_q &= a^{\lambda}b^{\mu+1}c^{\nu+1}a^{\lambda}c^{\nu}b^{\mu} \\ &= a^{2\lambda}b^{2\mu+1}c^{2\nu+1} \\ &= a^l b^m c^n = 1. \end{aligned}$$

In the *EEO* case

$$\begin{aligned} h_r &= b^{\mu}c^{\nu+1}a^{\lambda}c^{\nu} \\ &= a^{\lambda}b^{\mu}c^{2\nu+1} \\ &= a^{\lambda}b^{\mu}. \end{aligned}$$

Now both a^{λ} and b^{μ} are elements of order 2, so $h_r^2 = 1$. If G is cyclic then both a^{λ} and b^{μ} equal the unique element of order 2 in G, so $h_r = 1$. If G

is not cyclic then G must be a direct product of two cyclic groups since G is generated by a and b. Thus the subgroup $\{g \in G : g^2 = 1\}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the first case $h_r = 1$ in the second case it could be an element of order two. All other cases have similar proofs to one of the above.

Now we consider the mirrors. The subgroup $\mathrm{Cent}_G(\theta)$ is the subgroup of elements of order dividing 2. Thus $\mathrm{Cent}_G(\theta) = \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ unless $|G|$ is odd. If $|G|$ is even this group is isomorphic to $\mathbb{Z}_2$ if $|G|$ is cyclic and one of $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ if G is non-cyclic. Now in the even cases the centralizer of an oval will be a dihedral group with 2 or 4 elements. This gives us all the remaining entries in the table. ■

Commutative group algebra equations The group algebra equations 25 - 26 for an abelian group have a simple criterion for solvability as we now show.

Proposition 15 *Let $G = \langle x, y \rangle$ be a finite abelian group and suppose that x and y have orders s and t respectively. Let $\mathcal{R}$ be an arbitrary ring and $\mathcal{A}$ =$\mathcal{R}[G]$ be its group algebra. Suppose that α, β, $\eta \in \mathcal{A}$ satisfy the group algebra equations*

$$\eta(1-x) = \alpha, \tag{30}$$

$$\eta(1-y) = \beta. \tag{31}$$

Then, α, β, η must satisfy

$$\alpha(1 + x + x^2 + \cdots + x^{s-1}) = 0, \tag{32}$$

$$\beta(1 + y + y^2 + \cdots + y^{t-1}) = 0, \tag{33}$$

and

$$\alpha(1-y) = \beta(1-x). \tag{34}$$

Now let $\mathcal{R} = \mathbb{Q}$ and suppose that equations 32 - 34 hold. Then the equations 30 and 31 have a solution.

Proof. Since $(1-w)(1 + w + w^2 + \cdots + w^{u-1}) = 1 - w^u$ for any $w \in G$ and integer u then equations 32 and 33 hold for any group algebra. If G is abelian then

$$\alpha(1-y) = \eta(1-x)(1-y) = \eta(1-y)(1-x) = \beta(1-x)$$

and so equation 34 holds. Now suppose $\mathcal{R} = \mathbb{Q}$ and that the equations 32 - 34 hold. There are group algebra elements $\phi, \psi \in \mathbb{Q}[\langle x \rangle] \subset \mathcal{A}$ such that

$$1 = \phi(1 - x) + \psi(1 + x + x^2 + \cdots + x^{s-1}). \tag{35}$$

For, the polynomials $(1 - z)$ and $(1 + z + z^2 + \cdots + z^{s-1})$ are relatively prime and one can find ϕ, $\psi \in \mathbb{Q}[z]$ such that $1 = \phi(1-z)+\psi(1+z+z^2+\cdots+z^{s-1})$. (This is the point at which we need $\mathcal{R}$ be a field of characteristic 0.) Next identify the polynomial ring $\mathbb{Q}[z]$ as a subalgebra of the group algebra $\mathbb{Q}[\mathbb{Z}]$ and then apply the canonical homomorphism $\mathbb{Q}[\mathbb{Z}] \to \mathbb{Q}[\langle x \rangle]$. Let $\eta_0 = \alpha\phi$ and observe that

$$\begin{aligned}\alpha &= \alpha(\phi(1 - x) + \psi(1 + x + x^2 + \cdots + x^{s-1}))\\ &= \alpha\phi(1 - x) + \alpha(1 + x + x^2 + \cdots + x^{s-1})\psi\\ &= \eta_0(1 - x).\end{aligned}$$

Thus we have a "particular" solution to equation 30 If $\eta = \eta_0 + h$ is any other solution then $h(1 - x) = 0$. But then,

$$\begin{aligned}h &= h(\phi(1 - x) + \psi(1 + x + x^2 + \cdots + x^{s-1}))\\ &= h(1 - x)\phi + h\psi(1 + x + x^2 + \cdots + x^{s-1})\\ &= \rho(1 + x + x^2 + \cdots + x^{s-1})\end{aligned} \tag{36}$$

for some $\rho \in \mathcal{A}$, which can be arbitrary. Now we solve for ρ so that the second equation 31 is also satisfied. This means that $\beta = (\eta_0 + h)(1 - y)$ or

$$\begin{aligned}\beta - \eta_0(1 - y) &= h(1 - y)\\ &= \rho(1 + x + x^2 + \cdots + x^{s-1})(1 - y).\end{aligned}$$

Thus we need to show that $\omega = \beta - \eta_0(1 - y)$ has a factorization $\rho(1 + x + x^2 + \cdots + x^{s-1})(1 - y)$ for some $\rho \in \mathcal{A}$. If this can be proven then $\eta = \eta_0 + \rho(1 + x + x^2 + \cdots + x^{s-1})$ is a solution to our equations, as is easily verified. Note that by hypothesis ω satisfies:

$$\begin{aligned}\omega(1 - x) = (\beta - \eta_0(1 - y))(1 - x) =&\\ \beta(1 - x) - \eta_0(1 - x)(1 - y)&\\ = \beta(1 - x) - \alpha(1 - y)&\\ = 0,&\end{aligned}$$

and

$$\begin{aligned}
&\omega(1+y+y^2+\cdots+y^{t-1})\\
&=(\beta-\eta_0(1-y))(1+y+y^2+\cdots+y^{t-1})\\
&=\beta(1+y+y^2+\cdots+y^{t-1})+\eta_0(1-y)(1+y+y^2+\cdots+y^{t-1})\\
&=0.
\end{aligned}$$

Now, analogous to equation 35, there are $\phi', \psi' \in \mathbb{Q}[\langle y\rangle] \subset \mathcal{A}$ such that

$$1=\phi'(1-y)+\psi'(1+y+y^2+\cdots+y^{t-1}). \tag{37}$$

Using the equations 35 and 37 and arguing as in calculation 36 above we get

$$\begin{aligned}
\omega &= \omega\phi(1-x)+\omega\psi(1+x+x^2+\cdots+x^{s-1})\\
&=\omega\psi(1+x+x^2+\cdots+x^{s-1})\\
&=\omega\psi(1+x+x^2+\cdots+x^{s-1})\phi'(1-y)\\
&+\omega\psi(1+x+x^2+\cdots+x^{s-1})\psi'(1+y+y^2+\cdots+y^{t-1})\\
&=\omega\psi\phi'(1+x+x^2+\cdots+x^{s-1})(1-y)\\
&+\psi\psi'(1+x+x^2+\cdots+x^{s-1})\omega(1+y+y^2+\cdots+y^{t-1})\\
&=\omega\psi\phi'(1+x+x^2+\cdots+x^{s-1})(1-y).
\end{aligned}$$

We select $\rho=\omega\psi\phi'$ to complete our proof. ∎

The forgoing can be combined with Corollary 12 into a theorem on separability for abelian rotation groups.

Theorem 16 *Let G be an abelian rotation group with generating triple (a,b,c) and let φ be the reflection arising from the automorphism θ given in Proposition 13. Let*

$$\begin{aligned}
\eta(1-a^{-1})&=\gamma,\\
\eta(1-b)&=\delta,
\end{aligned}$$

be the separability equations arising from equations 25 - 28 in Corollary 12, where the γ, δ are chosen according to Table 5. Then φ is separating if and only if the following hold:

$$\begin{aligned}
\gamma(1+a+a^2+\cdots+a^{l-1})&=0,\\
\delta(1+b+b^2+\cdots+b^{m-1})&=0,\\
\delta(1-a^{-1})&=\gamma(1-b).
\end{aligned}$$

in the group algebra $\mathbb{Q}[G]$.

Example 1. Suppose that $G = \mathbb{Z}_n = \langle x \rangle$ and that $(l, m, n) = (n, n, n)$. Since $\sigma = \frac{n-1}{2}$ then n must be odd, say $n = 2\nu + 1$ and $\nu > 1$ since S is hyperbolic. The parity symbol of the action is OOO. We may assume that up to automorphisms that the generating triples (a, b, c) are of the form $(x, x^s, x^{-(s+1)})$ where $1 \leq s \leq n-1$, and $(s, n) = (s+1, n) = 1$. By the remarks following equations 7 and 8, all of p, q and r are conjugate, and hence that we must solve the system 25. From Table 1. $\mathcal{H}_q = 1$ and $x_{qp} = a^\nu$, $x_{qq} = 1$ and $x_{qr} = b^{\nu+1} = b^{-\nu}$. Thus we must solve:

$$\begin{aligned}\eta(1 - a^{-1}) &= \mathcal{H}_q(1 - a^\nu),\\ \eta(1 - b) &= \mathcal{H}_q(1 - b^{-\nu}).\end{aligned}$$

or

$$\begin{aligned}\eta(1 - x^{-1}) = \eta(1 - a^{-1}) &= (1 - x^\nu),\\ \eta(1 - x^s) = \eta(1 - b) &= (1 - x^{-s\nu}).\end{aligned}$$

Using the "cross multiplication" criterion, equation 34, from Theorem 16 we get the following necessary condition:

$$(1 - x^\nu)(1 - x^s) = (1 - x^{-s\nu})(1 - x^{-1})$$

or

$$1 - x^\nu - x^s + x^{s+\nu} = 1 - x^{-1} - x^{-s\nu} + x^{-1-s\nu}.$$

In order to achieve equality we must have (assuming $1 \neq -1$ in $\mathcal{R}$) one of $s \equiv -1 \bmod n$ or $\nu \equiv -1 \bmod n$. Neither of these conditions are possible since n cannot divide $s+1$ or $\nu+1$. Thus, none of these surfaces split at any mirror.

Example 2. Suppose that $G = \mathbb{Z}_{2k} = \langle x \rangle$ and that $(l, m, n) = (k, 2k, 2k)$. Then $\sigma = k - 1$, and $k \geq 3$ since S is hyperbolic. If k is odd then the parity of the action is OEE and $\lambda = \frac{k-1}{2}, \mu = \nu = k$, and if k is even then the parity is EEE with $\lambda = \frac{k}{2}, \mu = \nu = k$. We will consider only the generating triple $(a, b, c) = (x^{-2}, x, x)$. See [6] for the complete analysis, by other means, of all $(k, 2k, 2k)$ - actions of $\mathbb{Z}_{2k}$. In [6] it is shown that our chosen class of generating vectors is the only one to yield a splitting mirror.

First let us suppose that k is odd and hence that p and q are conjugate. We need to determine if r is conjugate to q. For this to happen we must have

$x = b = \theta(x^{-s})x^s = x^{2s}$, for some s, which is not possible. Thus, we need to solve the system 25. From the parity of the action, and Table 2 we see that $H_q = \langle c^\nu, b^\mu \rangle = \{1, x^k\}$ and hence $\mathcal{H}_q = 1 - x^k$ by Remark 5. From Table 4.a we get $x_{qp} = a^\lambda = x^{1-k}$, $x_{qq} = 1$ and $x_{qr} = 0$. Following the previous example we must solve:

$$\eta(1 - x^2) = (1 - x^k)(1 - x^{1-k}) = 1 + x - x^k - x^{k+1},$$
$$\eta(1 - x) = 1 - x^k.$$

The criteria of Theorem 16 are easily verified, but in this case is simple to construct a solution. By inspection we see that $\eta = 1 + x + x^2 + \cdots + x^{k-1}$ is a solution to the second equation. It is also easily seen to be a solution of the first equation.

Now suppose that k is even. Again, r is not conjugate to q but p is conjugate to q. For $x^2 = a^{-1} = \theta(x^{-s})x^s = x^{2s}$ does have solution. Note here that even though l is even p and q are still conjugate, though not by means of an element in $\langle a \rangle$. Now we must use the system 26. We may pick $g = x$ to be the conjugating element. We get $H_p = H_q = \{1, x^k\}$, $\mathcal{H}_p = \mathcal{H}_q = 1 - x^k$, $x_{pp} = 1$, and $x_{qq} = 1$, and all other $x_{ij} = 0$. Our equations become:

$$\eta(1 - x^2) = \varepsilon_g x(1 - x^k)(0 - 1) + (1 - x^k)(1 - 0) = (1 - x^k)(1 - \varepsilon_g x)$$
$$\eta(1 - x) = \varepsilon_g x(1 - x^k)(0 - 0) + (1 - x^k)(1 - 0) = 1 - x^k.$$

Instead of determining ε_g we are simply going to cheat and set $\mathcal{R} = \mathbb{F}_2$ to get the following:

$$\eta(1 + x^2) = 1 + x + x^k + x^{k+1},$$
$$\eta(1 + x) = 1 + x^k.$$

This may be solved as in the odd case.

Example 3. Suppose that $G = \mathbb{Z}_n \times \mathbb{Z}_n = \langle x, y : x^n = y^n = [x, y] = 1 \rangle$ and that $(l, m, n) = (n, n, n)$. Since $\sigma = \frac{(n-1)(n-2)}{2}$ then n may be any integer ≥ 4 if S is to be hyperbolic. It is easily shown that any pair of generators, both of order n, are equivalent under $\mathrm{Aut}(G)$. Thus, we may assume that $(a, b, c) = (x, y, y^{-1}x^{-1})$. Now suppose that $n = 2\nu + 1$ is odd so that $\nu > 1$. By the remarks following equations 7 and 8, all of p, q and r are conjugate. From Table 1 $\mathcal{H}_q = 1$ and $x_{qp} = a^\nu$, $x_{qq} = 1$ and $x_{qr} = b^{\nu+1}$. Thus we must solve

$$\eta(1-x^{-1}) = 1 - x^{\nu},$$
$$\eta(1-y) = 1 - y^{-\nu}.$$

Now, as above, we have the "cross-multiplication" criterion:

$$(1-x^{\nu})(1-y) = (1-y^{-\nu})(1-x^{-1})$$

or

$$1 - y - x^{\nu} + x^{\nu}y = 1 - x^{-1} - y^{-\nu} + x^{-1}y^{-\nu}.$$

We get a contradiction analogous to that in Example 1.

Now suppose that $n = 2\nu$ is even so that $\nu > 1$. The reflections p and q are conjugate if and only if there is a solution to $x^{-1} = x^{2s}y^{2t}$, which cannot happen. Similarly q is not conjugate to r. From the tables we get that $\mathrm{Stab}_G(\mathcal{O}_q) = H_q = \langle x^{\nu}, y^{\nu} \rangle$, $\mathcal{H}_q = (1 + x^{\nu}y^{\nu})(1 - x^{\nu}) = 1 - x^{\nu} - y^{\nu} + x^{\nu}y^{\nu}$ $= (1-x^{\nu})(1-y^{\nu})$, and that $x_{qp} = x_{qr} = 0$, $x_{qq} = 1$. Following our previous procedure we get the compatibility equations:

$$\eta(1-x^{-1}) = (1-x^{\nu})(1-y^{\nu}),$$
$$\eta(1-y) = (1-x^{\nu})(1-y^{\nu}).$$

and then

$$(1-x^{\nu})(1-y^{\nu})(1-y) = (1-x^{\nu})(1-y^{\nu})(1-x^{-1})$$

or

$$\begin{aligned} &1 - x^{\nu} - y^{\nu} - y + x^{\nu}y^{\nu} + x^{\nu}y + y^{\nu+1} - x^{\nu}y^{\nu+1} \\ &= 1 - x^{\nu} - y^{\nu} - x^{-1} + x^{\nu}y^{\nu} + x^{\nu-1} + x^{-1}y^{\nu} - x^{\nu-1}y^{\nu}. \end{aligned}$$

These expressions are different unless $\nu = 1$, and in this case we get the $(2,2,2)$ tiling of the sphere with separating reflections. In all other cases the reflections are not separating.

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